Quasi-local Mass and Momentum in General Relativity

Shing-Tung Yau

Harvard University

Stephen Hawking's 70th Birthday

University of Cambridge, Jan. 7, 2012

I met Stephen Hawking first time in 1978 when he invited me to Cambridge. I learnt a great deal from him through reading his works and through his students. In the past thirty years, I invited him to give a talk in the institute for advanced study in 1982, to give talks in China in 2002 and 2007, when I arranged him to meet the leaders of China. On his way to Beijing in 2007, I also arranged him to give talks in Hong Kong. I am glad that his fan may have been doubled because of these two trips to China.

Happy birthday, Stephen!


与科学大师对话
史蒂芬·霍金
——记

References

1. M.-T. Wang and S.-T. Yau, "Quasilocal mass in general relativity," *Phys. Rev. Lett.* 102:021101 (2009) [arXiv:0804.1174].

2. M.-T. Wang and S.-T. Yau, "Isometric embeddings into the Minkowski space and new quasi-local mass," *Comm. Math. Phys.* 288 (2009), no. 3, 919-942 [arXiv:0805.1370].

3. M.-T. Wang and S.-T. Yau, "Limit of quasilocal mass at spatial infinity," *Comm. Math. Phys.* 296 (2010), no.1, 271-283 [arXiv:0906.0200].

4. P. Chen, M.-T. Wang, and S.-T. Yau, "Evaluating quasilocal energy and solving optimal embedding equation at null infinity," *Comm. Math. Phys.* 308 (2011), no. 3, 845-863 [arXiv:1002.0927].

In this talk, I want to talk about quasilocal mass and momentum in general relativity: by the equivalence principle, it is not possible to define mass density in General relativity. So we look for quasilocal definition. What it means is that : given the metric and the first order deformation of a topologically sphere in space-time, we like to associate a four vector to measure the mass momentum within the region enclosed by this sphere.

Note that for a scalar wave on the Kerr background : the rotating Black hole, the energy density is negative within the ergosphere . This means definition of quasilocal mass is delicate .

The potential energy of a pair of gravitating particles depends on their separating distance. Hence the quasilocal energy depends on the gravitational field configuration.

In 1982, Penrose listed the search for a definition of such quasi-local mass as his number one problem in classical general relativity [in S.-T. Yau, *Seminar on Differential Geometry* (1982)].

There are many reasons to search for such a concept. Many important statements in general relativity make sense only with the presence of a good definition of quasi-local mass. For example, it allows us to talk about the binding energy of two bodies rotating around each other.

More importantly, a good definition of quasi-local mass should help us to control the dynamics of the gravitational field. Hopefully, this may be used to generalize the energy method in hyperbolic equations where difficulties were encountered even in the study of linearized stability of the Kerr metric.

There are various proposals for the definition of quasilocal mass:

S.W. Hawking, J. Math. Phys. (N.Y.) 9, 598 (1968).

R. Penrose, Proc. R. Soc. A 381, 53 (1982).

R. Bartnik, R. Phys. Rev. Lett. 62(20), 2346–2348 (1989).

J. D. Brown and J. W. York, Phys. Rev. D 47, 1407 (1993).

Properties we require for a valid definition:

(1) The ADM or Bondi mass should be recovered as spatial or null infinity is approached.

(2) The correct limits need be obtained when the surface converges to a point.

(3) Quasilocal mass must be nonnegative in general (under local energy condition) and zero when the ambient spacetime of the surface is the flat Minkowski spacetime.

(4) It should also behave well when the spacetime is spherically symmetric.

The hoop conjecture of Thorne (1972):

"Horizons form when and only when a mass m gets compacted into a region whose circumference in every direction is $C = 4\pi M$."

Schoen-Yau (1983):

If $\mu - |J| \geq \Lambda$ holds on a bounded region $\Omega \subset N$ for a spacelike hypersurface N in a spacetime, and $Rad(\Omega) \geq \sqrt{\frac{3}{2}}\frac{\pi}{\sqrt{\Lambda}}$, then N contains an apparent horizon.

I was unsatisfied with the result as it is too local, and does not involve boundary effect.

Yau (2001):

Suppose the mean curvature H of $\partial\Omega$ is strictly greater than $|tr_{\partial\Omega}(p)|$. Let $c = \min(H - |tr_{\partial\Omega}(p)|)$. If $Rad(\Omega) \geq \sqrt{\frac{3}{2}}\frac{\pi}{\sqrt{\Lambda}}$ where $\Lambda = \frac{2}{3}c^2 + \mu - |J|$, then Ω must admit an apparent horizon in its interior.

I want to replace mass of the region by quasilocal mass of the boundary surface, and the circumference of the hole by either the diameter as measured by the square root of area or some other type of length that can be described as circumference.

Gibbons (2009) reformulated the hoop conjecture in terms of the Birkhoff invariant or length of the shortest closed geodesic.

There were many attempts to give the definition of quasi-local mass. We shall use an approach which seems to be most promising.

Recall that for a Lorenztian manifold M with boundary ∂M, the action in general relativity should be

$$I(g,\Phi) = \int_M \left(\frac{1}{16\pi} R + L(g,\Phi) \right) + \frac{1}{8\pi} \int_{\partial M} K$$

where K is the trace of the second fundamental form of ∂M.

The last term is needed to give rise to the right variational equation if we fix the metric and the matter field on the boundary.

If we demand that a certain background (g_0, Φ_0) is a static solution to the field equation, we replace I by

$$I(g, \Phi) - I(g_0, \Phi_0) \ .$$

Hence, for flat spacetime background, we use

$$\int_M \left(\frac{1}{16\pi} R + L(g, \Phi) \right) + \frac{1}{8\pi} \int_{\partial M} (K - K_0) \ .$$

Suppose we take a family of space-like surface Σ_t and a time-like vector field t such that $t^\mu \nabla_\mu t = 1$. We can write

$$t^\mu = N\, n^\mu + N^\mu$$

where n^μ is the normal to Σ_t,

N is called the lapse function,

N^μ is called the shift vector.

In this notation,

$$I(g,\Phi) = \int N\, dt \left[\frac{1}{16\pi} \int_{\Sigma_t} \left(R + p_{\mu\nu} p^{\mu\nu} - p^2 + 16\pi L \right) + \frac{1}{8\pi} \int_{S_t^2} {}^2K \right]$$

where $p_{\mu\nu}$ is the second fundamental form of Σ_t and p is its trace, ${}^2K$ is the mean curvature of $\partial\Sigma_t = S_t$.

If one introduces the canonical momenta $k^{\mu\nu}$, k conjugate to ${}^3g_{\mu\nu}, \Phi$, we can rewrite the action to be

$$\int dt \int_{\Sigma_t} \left(k^{\mu\nu} \dot{g}_{\mu\nu} + k\dot{\Phi} - N\,\mathcal{H} - N^\mu\,\mathcal{H}_\mu \right) + \frac{1}{8\pi} \int_{S_t} \left(N\,{}^2K - N^\mu p_{\mu\nu} r^\nu \right)$$

where $\mathcal{H}$ is the Hamiltonian constraint

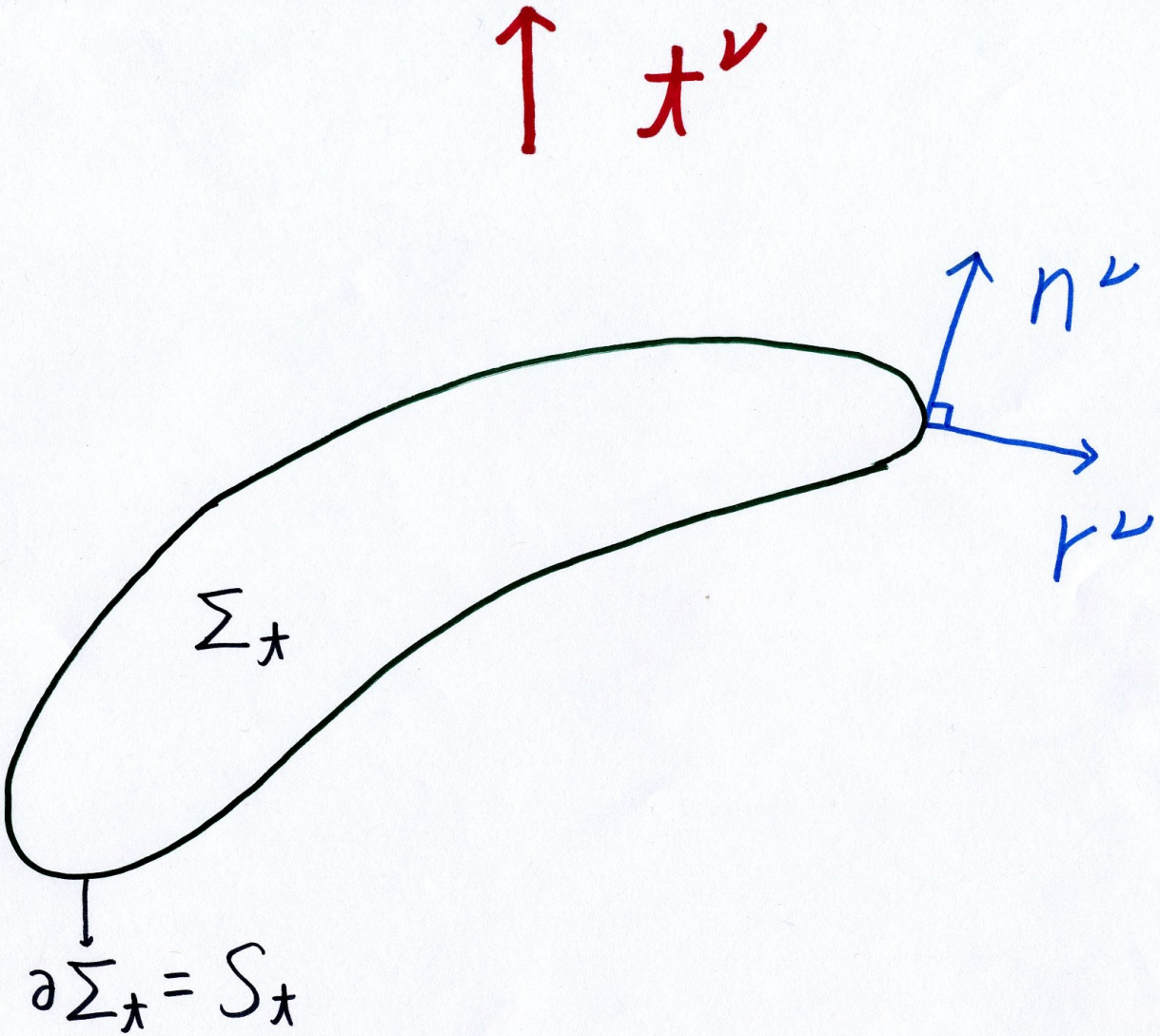
$$T_{00} - \frac{1}{2}\left(R - p_{\mu\nu} p^{\mu\nu} + p^2 \right)$$

and $\mathcal{H}_\mu$ is the momentum constraint

$$T_{0\mu} - p_{\mu\nu,\nu} + p_{,\mu} \ .$$

Note that $\mathcal{H} = 0$ and $\mathcal{H}_\mu = 0$ when the equation of motion is satisfied.

r^ν is spacelike unit normal to S_t and tangent to Σ_t.


t^ν
n^ν
r^ν
Σ_t
$\partial\Sigma_t = S_t$

The Hamiltonian is then derived to be

$$H = \int_{\Sigma_t} (N\,\mathcal{H} + N^\mu\,\mathcal{H}_\mu) - \frac{1}{8\pi}\int_{S_t} \left(N\,{}^2K - N^\mu p_{\mu\nu} r^\nu\right) \ .$$

If we take the background so that $p_{\mu\nu} = 0$, we see that the Hamiltonian relative to the background is given by

$$\int_{\Sigma_t} (N\,\mathcal{H} + N^\mu\,\mathcal{H}_\mu) - \frac{1}{8\pi}\int_{S_t} \left(N({}^2K - {}^2K_0) - N^\mu p_{\mu\nu} r^\nu\right) \ .$$

Hence associated to each time-like vector field t, we have the physical Hamiltonian

$$-\frac{1}{8\pi}\int_{S_t}\left(N({}^2K - {}^2K_0) - N^\mu p_{\mu\nu} r^\nu\right) \ .$$

This expression was derived by Brown-York and Hawking-Horowitz.

They proposed to simply choose $N = 1, N^\mu = 0$ for the definition of quasi-local mass. In general, the definition does not give positivity except in the time symmetric case ($p_{\mu\nu} = 0$) which was proved by Shi-Tam.

The definition of Brown-York is gauge dependent. Liu-Yau defined a gauge independent mass to be

$$-\frac{1}{8\pi}\int_S \left(\sqrt{({}^2K)^2-(tr_S p)^2}-{}^2K_0\right)$$

and proved that it is positive whenever the mean curvature vector of S is space-like and the Gauss curvature is positive.

The proof combined arguments of Schoen-Yau and Witten. We needed to handle metrics where the mean curvature may jump along the boundary. The discontinuity of the Dirac spinor required nontrivial analysis.

$\sqrt{({}^2K)^2-(tr_S p)^2}$ is the Lorentian norm of the mean curvature vector

$$H=-{}^2K\, r^\nu+(tr_S p)n^\nu.$$

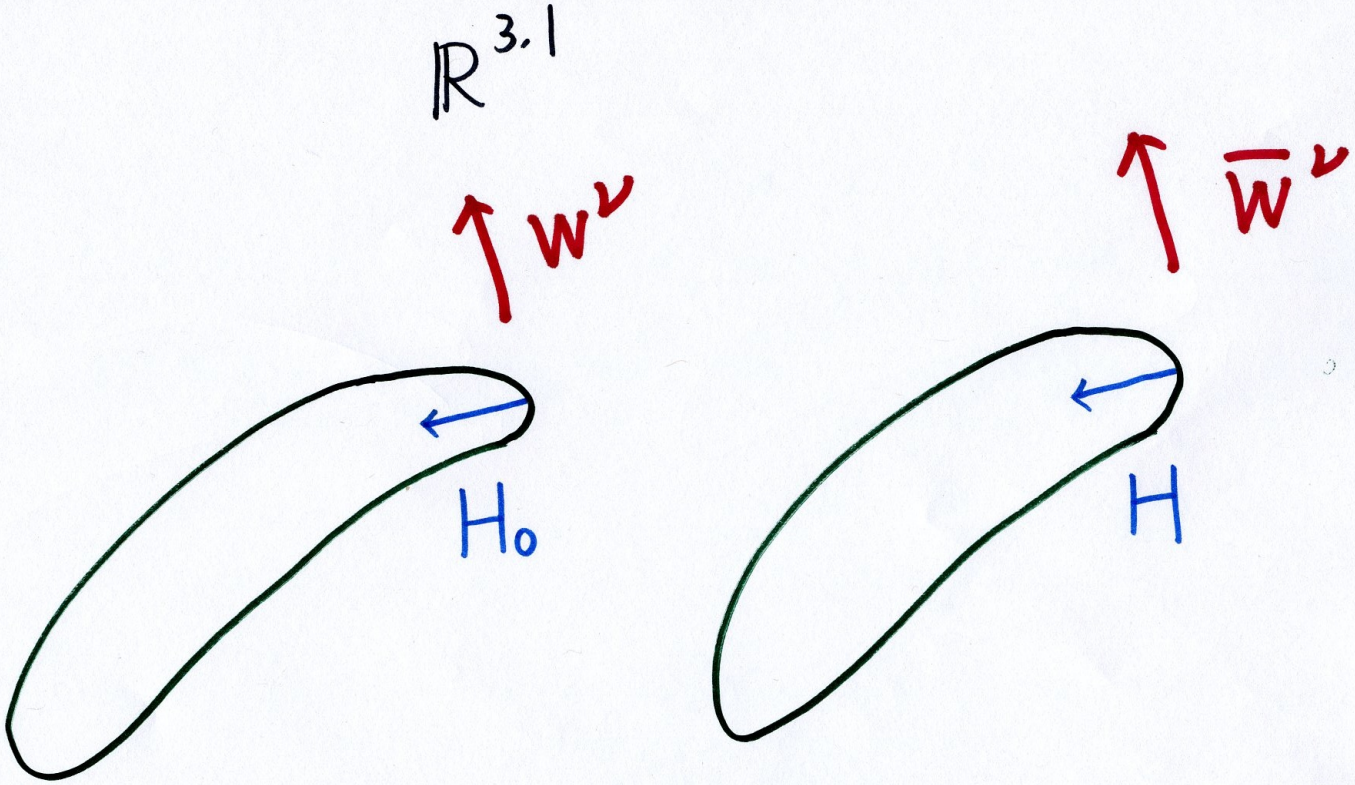
Let me now describe the work that I did with Mu-Tao Wang.

Given a surface S, we assume that its mean curvature is positive. We embed S isometrically into $\mathbb{R}^{3,1}$.

Given any constant unit future time-like vector w (observer) in $\mathbb{R}^{3,1}$, we can define a future directed time-like vector field $\overline{w}$ along S by requiring

$$\langle H_0, w \rangle \;=\; \langle H, \overline{w} \rangle$$

where H_0 is the mean curvature vector of S in $\mathbb{R}^{3,1}$
and H is the mean curvature vector of S in spacetime.



$$\langle H_0, w \rangle = \langle H, \bar{w} \rangle$$

$$w^\nu = N n^\nu + N^\nu \qquad \bar{w}^\nu = N \bar{n}^\nu + N^\nu$$

Note that given any surface S in $\mathbb{R}^{3,1}$ and a constant future time-like unit vector w^ν, there exists a canonical gauge n^μ (future time-like unit normal along S) such that

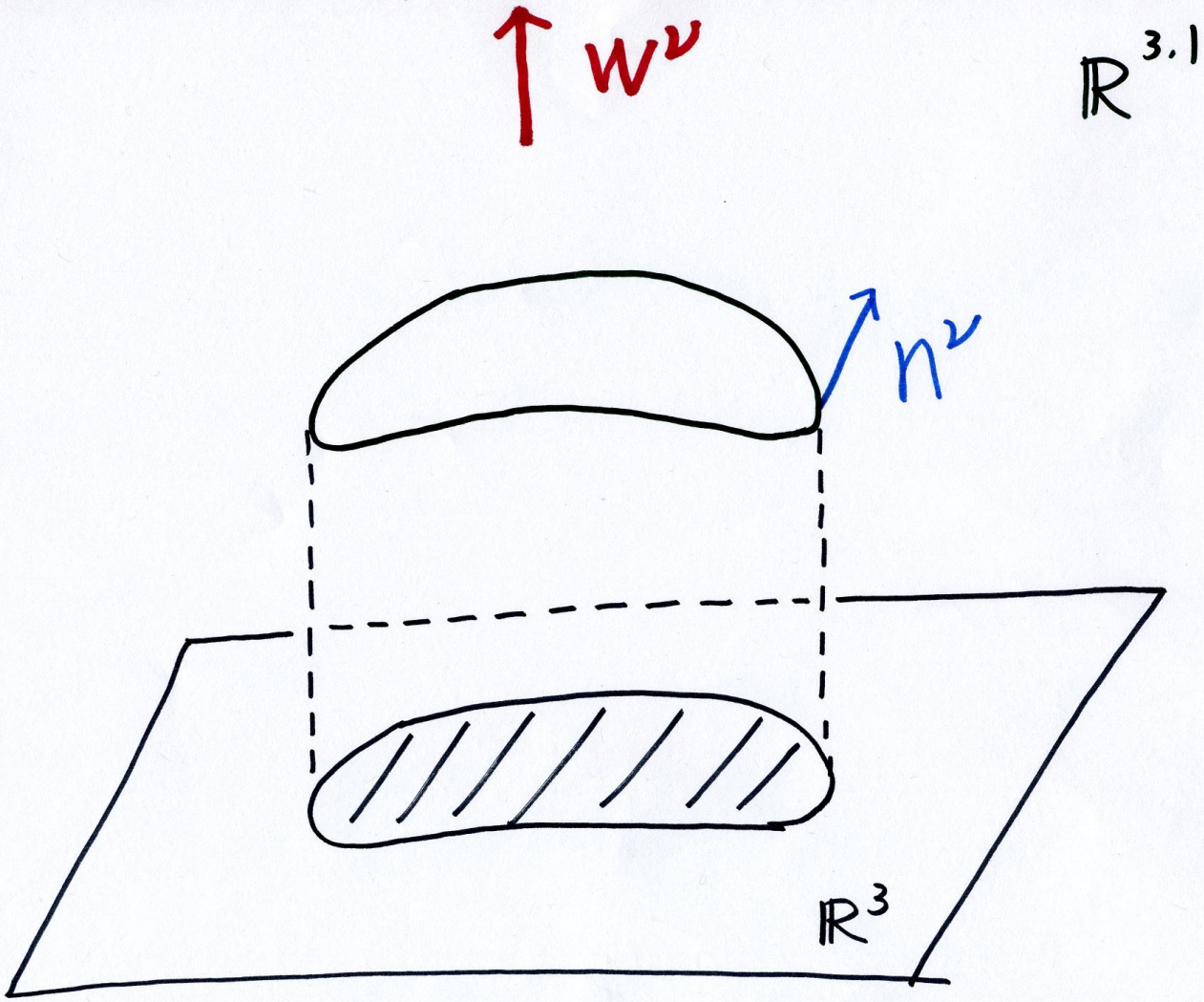
$$\int_S N^2 K_0 + N^\mu (p_0)_{\mu\nu} r^\nu$$

is equal to the total mean curvature of $\hat{S}$, the projection of S onto the orthogonal complement of w^μ ,

$w^\mu = N\, n^\mu + N^\mu$,

r^μ is the space-like unit normal orthogonal to n^μ ,

p_0 is the second fundamental form calculated by the three surface defined by S and r^μ.


w^ν
$\mathbb{R}^{3,1}$
n^ν
$\mathbb{R}^3$

From the matching condition and the correspondence $(w^\mu, n^\mu) \to (\overline{w}^\mu, \overline{n}^\mu)$, we can define a similar quantity from the data in spacetime

$$\int_S N\,{}^2\overline{K} + N^\mu(\overline{p})_{\mu\nu}\overline{r}^\nu \ .$$

We write $E(w)$ to be

$$8\pi E(w) = \int_S N\,{}^2\overline{K} + N^\mu(\overline{p})_{\mu\nu}\overline{r}^\nu - \int_S N\,{}^2K_0 + N^\mu(p_0)_{\mu\nu}r^\nu$$

and define the quasi-local mass to be

$$\inf E(w)$$

where the infimum is taken among all isometric embeddings into $\mathbb{R}^{3,1}$ and timelike unit constant vector $w \in \mathbb{R}^{3,1}$.

The Euler-Lagrange equation for minimizing $E(w)$ is

$$div_S\left(\frac{\nabla\tau}{\sqrt{1+|\nabla\tau|^2}}\cosh\theta|H| - \nabla\theta - V\right)$$
$$-\left(\widehat{H}\hat{\sigma}^{ab} - \hat{\sigma}^{ac}\hat{\sigma}^{bd}\hat{h}_{cd}\right)\frac{\nabla_b\nabla_a\tau}{\sqrt{1+|\nabla\tau|^2}} = 0$$

where $\sinh\theta = \frac{-\Delta\tau}{|H|\sqrt{1+|\nabla\tau|^2}}$, V is the tangent vector on Σ that is dual to the connection one-form $\langle\nabla^N_{(\cdot)}\frac{J}{|H|}, \frac{H}{|H|}\rangle$ and $\hat{\sigma}$, $\widehat{H}$ and $\hat{h}$ are the induced metric, mean curvature and second fundamental form of $\hat{S}$ in $\mathbb{R}^3$.

In general, the above equation should have an unique solution τ.

We prove that $E(w)$ is non-negative among admissible isometric embedding into Minkowski space.

In the proof, we use the techniques and results of the positivity of the Liu-Yau quasi-local mass. First we prove an inequality about total mean curvature for solutions of Jang's equation. Then we prove positivity of $E(W)$ by comparing the defined mass to a similar quantity defined on the graph of the solution to Jang's equation whose boundary condition is the given time function.

In summary, given a closed space-like 2-surface in spacetime whose mean curvature vector is space-like, we associate an energy-momentum four-vector to it that depends only on the first fundamental form, the mean curvature vector and the connection of the normal bundle with the properties

1. It is Lorentzian invariant;

2. It is trivial for surfaces sitting in Minkowski spacetime and future time-like for surfaces in spacetime which satisfies the local energy condition.

Spherical symmetric spacetime are foliated by the orbits of $SU(2)$. We can define a function on the spacetime by associating to its orbit the area $4\pi r^2$.

The mean curvature vector of the orbit is

$$-\frac{2}{r}\nabla r$$

where ∇ is with respect to the quotient Lorentzian $(1,1)$ metric.

If this vector is space-like, the quasi-local mass of this orbit sphere is

$$M = r\left(1 - |\nabla r|\right) \ .$$

Note that in 1964, Misner and Sharp defined a mass

$$m = \frac{r}{2}\left(1 - |\nabla r|^2\right)$$

which is the same as the Hawking mass (1968)

$$\sqrt{\frac{A}{16\pi}}\left(1 - \frac{1}{16\pi}\int_S |H|^2\right) \ .$$

The relation with our mass is

$$m = M - \frac{M^2}{2r} \ .$$

From the formula of quasi-local mass, which we proved to be positive, we derived a corollary that the mass m (Hawking mass) is also positive. (This was proved by Christodoulou (1995) under extra assumptions.)

Note that

$$\frac{1}{2} M \leq m \leq M \ .$$

On the apparent horizon $M = 2m$,
and at space-like infinity $M = m$.

Hence our quasi-local mass is equivalent to the standard definition in the case of spherically symmetric spacetime.

In the spatial direction, the Hawking mass is monotonically increasing along the inverse mean curvature flow (Geroch) and this is important in Huisken-Ilmanen's work.

The quasi-local mass is not monotonically increasing in this sense. However, the spherical symmetric case indicates that such property may still hold, up to a constant depending on the initial surface.

In the future time-like or null direction, the quasi-local mass is expected to decrease up to a constant depending on the initial surface if we choose the equation of motion for the 2-surfaces carefully.

In the case when $p_{ij} \equiv 0$, there is also a definition of quasi-local mass by Bartnik which is obtained by minimizing the ADM mass among all asymptotically flat extension of the data which does not contain an apparent horizon and which extends the original data.

Our quasi-local mass also satisfies the following important properties:

3. When we consider a sequence of spheres on an asymptotically flat space-like hypersurface, in the limit, the quasi-local mass (energy-momentum) is the same as the well-understood ADM mass (energy-momentum);

4. When we take the limit along a null cone, we obtain the Bondi mass(energy-momentum).

5. When we take the limit approaching a point along null geodesics, we recover the energy-momentum tensor of matter density when matter is present, and the Bel-Robinson tensor in vacuum.

These properties of the quasi-local mass is likely to characterize the definition of quasi-local mass, i.e. any quasi-local mass that satisfies all the above five properties may be equivalent to the one that we have defined.

Strictly speaking, we associate each closed surface not a four-vector, but a function defined on the light cone of the Minkowski spacetime. Note that if this function is linear, the function can be identified as a four-vector.

It is a remarkable fact that for the sequence of spheres converging to spatial infinity, this function becomes linear, and the four-vector is defined and is the ADM four-vector that is commonly used in asymptotically flat spacetime. For a sequence of spheres converging to null infinity in Bondi coordinate, the four vector is the Bondi-Sachs four-vector.

It is a delicate problem to compute the limit of our quasi-local mass at null infinity and spatial infinity. The main difficulties are the following:

(i) The function associated to a closed surface is in non-linear in general;

(ii) One has to solve the Euler-Lagrange equation for energy minimization.

For (i), the following observation is useful:

For a family of surfaces Σ_r and a family of isometric embeddings X_r of Σ_r into $\mathbb{R}^{3,1}$, the limit of quasi-local mass is a linear function under the following general assumption that the mean curvature vectors are comparable in the sense

$$\lim_{r\to\infty} \frac{|H_0|}{|H|} = 1$$

where H is the the spacelike mean curvature vector of Σ_r in N and H_0 is that in the image of X_r in $\mathbb{R}^{3,1}$.

Under the comparable assumption of mean curvature, the limit of our quasi-local mass with respect to a constant future time-like vector $T_0 \in \mathbb{R}^{3,1}$ is given by

$$\lim_{r\to\infty} \frac{1}{8\pi} \int_{\Sigma_r} \Big[- \langle T_0, \frac{J_0}{|H_0|} \rangle (|H_0| - |H|) \\ - \langle \nabla^{\mathbb{R}^{3,1}}_{\nabla\tau} \frac{J_0}{|H_0|}, \frac{H_0}{|H_0|} \rangle + \langle \nabla^{N}_{\nabla\tau} \frac{J}{|H|}, \frac{H}{|H|} \rangle \Big] \, d\Sigma_r$$

where $\tau = -\langle T_0, X_r \rangle$ is the time function with respect to T_0.

This expression is linear in T_0 and defines an energy-momentum four-vector at infinity.

At the spatial infinity of an asymptotically flat spacetime, the limit of our quasi-local mass is

$$\lim_{r\to\infty} \frac{1}{8\pi} \int_{\Sigma_r} (|H_0| - |H|)\, d\Sigma_r = M_{ADM}$$

$$\lim_{r\to\infty} \frac{1}{8\pi} \int_{\Sigma_r} \langle \nabla^N_{-\nabla X_i} \frac{J}{|H|}, \frac{H}{|H|} \rangle d\Sigma_r = P_i$$

where $\begin{pmatrix} M \\ P_i \end{pmatrix}$ is the ADM energy-momentum four-vector, assuming the embeddings X_r into $\mathbb{R}^3$ inside $\mathbb{R}^{3,1}$.

At the null infinity, the limit of quasi-local mass was found by Chen-Wang-Yau to recover the Bondi-Sachs energy-momentum four-vector.

On a null cone $w = c$ as r goes to infinity, the limit of the quasi-local mass is

$$\lim_{r\to\infty} \frac{1}{8\pi} \int_{\Sigma_r} (|H_0| - |H|) d\Sigma_r = \frac{1}{8\pi} \int_{S^2} 2m \, dS^2$$

$$\lim_{r\to\infty} \frac{1}{8\pi} \int_{\Sigma_r} \langle \nabla^N_{-\nabla X_i} \frac{J}{|H|}, \frac{H}{|H|} \rangle d\Sigma_r = \frac{1}{8\pi} \int_{S^2} 2mX_i \, dS^2$$

where $(X_1, X_2, X_3) = (\sin\theta \sin\phi, \sin\theta \cos\phi, \cos\theta)$.

The following two properties are important for solving the Euler-Lagrange equation for energy minimization:

(a) The limit of quasi-local mass is stable under $O(1)$ perturbation of the embedding;

(b) The four-vector obtained is equivariant with respect to Lorentzian transformations acting on X_r.

We observe that momentum is an obstruction to solving the Euler-Lagrange equation near a boosted totally geodesics slice in $\mathbb{R}^{3,1}$. Using (b), we find a solution by boosting the isometric embedding according to the energy-momentum at infinity. Then the limit of quasi-local mass is computed using (a) and (b).

In evaluating the small sphere limit of the quasilocal energy, we pick a point p in spacetime and consider C_p the future light cone generated by future null geodesics from p. For any future directed timelike vector e_0 at p, we define the affine parameter r along C_p with respect to e_0. Let S_r be the level set of the affine parameter r on C_p.

We solve the optimal isometric equation and find a family of isometric embedding X_r of S_r which locally minimizes the quasi-local energy.

With respect to X_r, the quasilocal energy is again linearized and is equal to

$$\frac{4\pi}{3} r^3 T(e_0, \cdot) + O(r^4)$$

which is the expected limit.

In the vacuum case, i.e. $T = 0$, the limit is non-linear with the linear term equal to

$$\frac{1}{90} r^5 Q(e_0, e_0, e_0, \cdot) + O(r^6)$$

with an additional positive correction term in quadratic expression of the Weyl curvature.

The linear part consists of the Bel-Robinson tensor and is precisely the small-sphere limit of the Hawking mass which was computed by Horowitz and Schmidt.

The Bel-Robinson tensor satisfies conservation law and is an important tool in studying the dynamics of Einstein's equation, such as the stability of the Minkowski space (Christodoulou-Klainerman) and the formation of trapped surface in vacuum (Christodoulou)