The nuts and bolts of AdS/CFT

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The State of the Universe, Cambridge
7 January 2012

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Nut Charge, Anti-de Sitter Space and Entropy

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(4 September 1998)

Abstract

It has been proposed that spacetimes with a $U(1)$ isometry group have contributions to the entropy from Misner strings as well as from the area of $d-2$ dimensional fixed point sets. In this paper we test this proposal by constructing Taub-Nut-AdS and Taub-Bolt-AdS solutions which are examples of a new class of asymptotically locally anti-de Sitter spaces. We find that with the additional contribution from the Misner strings, we exactly reproduce the entropy calculated from the action by the usual thermodynamic relations. This entropy has the right parameter dependence to agree with the entropy of a conformal field theory on the boundary, which is a squashed three-sphere, at least in the limit of large squashing. However the conformal field theory and the normalisation of the entropy remain to be determined.

04.70.Dy, 04.20.-q

A closely related paper:

Large N Phases, Gravitational Instantons and the Nuts and Bolts of AdS Holography

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Abstract

Recent results in the literature concerning holography indicate that the thermodynamics of quantum gravity (at least with a negative cosmological constant) can be modeled by the large N thermodynamics of quantum field theory. We emphasize that this suggests a completely unitary evolution of processes in quantum gravity, including black hole formation and decay; and even more extreme examples involving topology change. As concrete examples which show that this correspondence holds even when the space–time is only *locally* asymptotically AdS, we compute the thermodynamical phase structure of the AdS–Taub–NUT and AdS–Taub–Bolt spacetimes, and compare them to a 2+1 dimensional conformal field theory (at large N) compactified on a squashed three sphere, and on the twisted plane.

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The nuts and bolts of supersymmetric gauge theories on biaxially squashed three-spheres

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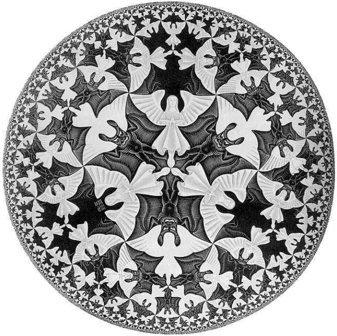
Abstract

We present the gravity dual to a class of three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories on a biaxially squashed three-sphere, with a non-trivial background gauge field. This is described by a 1/2 BPS Euclidean solution of four-dimensional $\mathcal{N} = 2$ gauged supergravity, consisting of a Taub-NUT-AdS metric with a non-trivial instanton for the graviphoton field. The holographic free energy of this solution agrees precisely with the large N limit of the free energy obtained from the localized partition function of a class of Chern-Simons quiver gauge theories. We also discuss a different supersymmetric solution, whose boundary is a biaxially squashed Lens space $S^3/\mathbb{Z}_2$ with a topologically non-trivial background gauge field. This metric is of Eguchi-Hanson-AdS type, although it is not Einstein, and has a single unit of gauge field flux through the S^2 cycle.



Taub-NUT-AdS is a one-parameter family of negatively curved complete Einstein metrics on $\mathbb{R}^4$:

$$R_{ij} = -3\, g_{ij} .$$

$$ds^2 = \frac{dr^2}{\Omega(r)} + 4s^2\Omega(r)(d\psi + \cos\theta d\phi)^2 + (r^2 - s^2)(d\theta^2 + \sin^2\theta d\phi^2) ,$$

where

$$\Omega(r) = \frac{(r-s)\,[1 + (r-s)(r+3s)]}{r+s} .$$

- The origin of $\mathbb{R}^4$ is at $r = s$ (the "nut"), where s is a parameter.
- At large r the metric is asymptotically locally AdS_4 with a "squashed" S^3 boundary:

$$ds^2 \simeq \frac{dr^2}{r^2} + r^2 \left[4s^2(d\psi + \cos\theta d\phi)^2 + (d\theta^2 + \sin^2\theta d\phi^2)\right] .$$

- When $s = 1/2$ the metric is (Euclidean) AdS_4 with round S^3 boundary.

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- At large $\mathbf{r}$ the metric is asymptotically locally AdS$_4$ with a "squashed" $\mathbf{S^3}$ boundary:

$$\mathbf{ds^2 \simeq \frac{dr^2}{r^2} + r^2 \left[4s^2(d\psi + \cos\theta d\phi)^2 + (d\theta^2 + \sin^2\theta d\phi^2) \right]} .$$

- When $\mathbf{s = 1/2}$ the metric is (Euclidean) AdS$_4$ with round $\mathbf{S^3}$ boundary.

According to the AdS/CFT correspondence, quantum gravity in this background has a dual description in terms of putting a CFT on the squashed $\mathbf{S^3}$ boundary:

> *"One can ask whether the partition function of a CFT on the boundary is related to the action of these solutions" (which determines the gravitational partition function). [Hawking-Hunter-Page]*

The immediate problem, though, is: what is this CFT?

> *"Another issue that has to be resolved is what CFT to use on the squashed* $\mathbf{S^3}$*. Here we are on shakier ground... On the three-dimensional boundary of* AdS_4 *Yang-Mills theory is not conformally invariant. The best we can do is calculate the determinants of free fields on the squashed* $\mathbf{S^3}$ *and see if they have the same dependence on the squashing as the action." [Hawking-Hunter-Page]*

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In fact all these questions may be answered, but to do so I have to cheat slightly and consider not pure gravity in AdS_4, but rather embed into M-theory.

$AdS_4 \times S^7$ is a supersymmetric solution to $d = 11$ supergravity. As well as the metric, there is a four-form field strength G with

$$\mathbb{N} \ni N = \frac{1}{(2\pi \ell_p)^6} \int_{S^7} *G \, .$$

- AdS/CFT dual to the three-dimensional theory on N M2-branes.
- The AdS radius is $R \propto N^{1/6} \ell_p$, so we can use supergravity only in the large N limit.

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What is this three-dimensional CFT on N M2-branes?

One answer is that it is the strong coupling limit of the D2-brane worldvolume theory = maximally supersymmetric $U(N)$ Yang-Mills theory. This is not conformally invariant, but is believed to flow in the IR to the M2-brane CFT.

A better description was found by Aharony-Bergman-Jafferis-Maldacena in 2008, using a string theory dual construction.

It is a $\mathbf{U(N)} \times \mathbf{U(N)}$ Chern-Simons gauge theory, with two matter fields transforming in the $\mathbf{(N, \overline{N})}$ representation, and two in the conjugate $\mathbf{(\overline{N}, N)}$ representation, interacting via a sextic potential.

$$\mathbf{S_{Chern-Simons}} = \sum_{i=1}^{2} \frac{(-1)^i}{4\pi} \int \mathrm{Tr}\left[A_i \wedge dA_i + \frac{2}{3} A_i^3\right] .$$

The theory is superconformally invariant, with manifest $\mathcal{N} = \mathbf{6}$ supersymmetry.

[For $\mathbf{N = 1}$ the matter fields are the 8 real transverse scalars to the M2-brane.]

This theory is conformally invariant, so there is no problem with putting it on the round $\mathbf{S^3}$. But what about our squashed $\mathbf{S^3}$?

This was recently answered by Imamura-Yokoyama (Sep 2011). They showed that *any* $\mathcal{N} = \mathbf{2}$ Chern-Simons-Yang-Mills gauge theory, with arbitrary matter content and interactions, can be put on the squashed $\mathbf{S^3}$, *preserving supersymmetry*.

Key to this is that we must also turn on a background $\mathbf{U(1)}$ gauge field $\mathbf{A^{(3)}}$ on the squashed $\mathbf{S^3}$:

$$\mathbf{A^{(3)}} \;=\; \mathbf{s\sqrt{1-4s^2}(d\psi + \cos\theta d\phi)} \;.$$

Physically, this is gauging the $\mathbf{U(1)_R}$ symmetry of the theory. [The Killing spinors are charged under this gauge field.]

Imamura-Yokoyama didn't appreciate this had anything to do with Taub-NUT-AdS.

Their motivation was that the partition function of this theory (following earlier work of Pestun, Kapustin-Willett-Yaakov, Hama-Hosomichi-Lee) is *exactly computable*:

$$\mathbf{Z} = \int_{\text{all fields}} \mathbf{e}^{-\mathbf{S}} \equiv \int_{\mathcal{Q}-\text{invariant fields}} \mathbf{e}^{-\mathbf{S}} \cdot (\text{one-loop determinant}) \, .$$

This is a form of *fixed point theorem*: $\mathcal{Q}$ is a supercharge, generating a supersymmetry variation of the theory.

It turns out this reduces the infinite-dimensional functional integral to a *finite-dimensional* integral [over the zero modes of the scalars in the vector multiplet].

For the ABJM theory this partition function is exactly equal to

$$\mathbf{Z[s;N]} \;=\; \prod_{n=1}^{N} \int_{-\infty}^{\infty} dx_n \int_{-\infty}^{\infty} dy_n \exp\left[4\pi i s^2 \sum_{n=1}^{N} (x_n^2 - y_n^2)\right] \times \frac{\prod_{n\neq m} s_b\left[2s(x_n - x_m - i)\right] s_b\left[2s(y_n - y_m - i)\right]}{\prod_{n,m} s_b^2\left[2s(x_n - y_m - \frac{i}{2})\right] s_b^2\left[2s(y_n - x_m - \frac{i}{2})\right]} ,$$

where $\mathbf{b = b(s) \equiv 2s + i\sqrt{1-4s^2}}$ and $\mathbf{s_b}$ is the double sine function, also known as the *quantum dilogarithm function*. The latter arises from zeta function regularization of the one-loop determinants.

To compare to the gravity dual, we must compute this in the $\mathbf{N} \to \infty$ limit.

This can be done using matrix integral methods developed over the last year by Klebanov et al, Martelli-JFS, Cheon-Kim-Kim.

The answer is that on the squashed $\mathbf{S}^3$

$$-\log Z[s; N] \;=\; \frac{4\sqrt{2}\pi s^2}{3} N^{3/2} + o(N^{3/2}) \; .$$

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We can now try to match this to the Euclidean quantum gravity result, which according to AdS/CFT should exactly reproduce the leading $\mathbf{N}^{3/2}$ term above.

The key difference to Hawking-Hunter-Page is that the dual field theory is supersymmetric, plus we turned on a boundary background gauge field.

Martelli and I made the Taub-NUT-AdS solution *supersymmetric* by turning on a specific finite action self-dual $\mathbf{U(1)}$ gauge field $\mathbf{A}$; more precisely, this is a $1/2$ BPS solution to $\mathbf{d = 4}$, $\mathcal{N} = \mathbf{2}$ gauged supergravity.

This $\mathbf{U(1)}$ gauge field $\mathbf{A}$ in Taub-NUT-AdS restricts on the boundary to the gauge field $\mathbf{A^{(3)}}$ of Imamura-Yokoyama.

Moreover, this four-dimensional solution uplifts to an *exact* solution of $\mathbf{d = 11}$ supergravity, using results of Gauntlett-Varela.

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$$
\begin{aligned}
\mathrm{ds}^2_{11} &= \mathrm{R}^2 \left[\mathrm{ds}^2_{\mathrm{TN-AdS}} + (\eta + \tfrac{1}{2}\mathrm{A})^2 + \mathrm{ds}^2_{\mathrm{CP}^3} \right] , \\
\mathrm{G} &= \mathrm{R}^3 \left[\tfrac{3}{8}\mathrm{vol}_{\mathrm{TN-AdS}} - *_4 \mathrm{dA} \wedge \mathrm{d}\eta \right] .
\end{aligned}
$$

The Euclidean quantum gravity partition function is $\mathbf{Z_{gravity}} = \mathbf{e^{-S_{regularized}}}$, where the regularized on-shell action for the solution is

$$\mathbf{S_{regularized}} \;=\; \mathbf{S^{bulk}_{Einstein-Hilbert} + S^{bulk}_{Maxwell} + S^{boundary}_{Gibbons-Hawking} + S^{boundary}_{counterterm}}\;.$$

Each term is a complicated function of $\mathbf{s}$, but the final answer simplifies to

$$\mathbf{S_{regularized}} \;=\; \frac{\mathbf{2\pi s^2}}{\mathbf{G_N}}\;.$$

The four-dimensional Newton constant $\dfrac{\mathbf{1}}{\mathbf{G_N}} = \dfrac{\mathbf{2\sqrt{2}\,N^{3/2}}}{\mathbf{3}}$ is computed via Kaluza-Klein.

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We thus find exact agreement between field theory and gravity calculations, as a function of the squashing parameter **s**, *including the precise numerical coefficient.*

This result generalizes in many directions. For example, one can put *different* CFTs on the squashed $\mathbf{S^3}$.

One way to engineer such field theories is to put the $\mathbf{N}$ M2-branes at a *Calabi-Yau singularity*. This has led to rich families of examples.

Martelli-JFS: Take the CFT to be a $\mathbf{U(N)} \times \mathbf{U(N)} \times \mathbf{U(N)}$ Chern-Simons theory with matter in the $\mathbf{(N, \overline{N}, 1)}$, $\mathbf{(\overline{N}, 1, N)}$, $\mathbf{(1, N, \overline{N})}$, plus conjugate representations, an adjoint matter field for *one* of the $\mathbf{U(N)}$ factors, plus appropriate potential. We computed

$$-\log Z[s;N] \;=\; \frac{32\pi s^2 N^{3/2}}{3}(2-\Delta)(1-\Delta)\sqrt{\frac{\Delta}{4-3\Delta}} + o(N^{3/2}) \; ,$$

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$$-\log Z[s;N] \;=\; \frac{32\pi s^2 N^{3/2}}{3}(2-\Delta)(1-\Delta)\sqrt{\frac{\Delta}{4-3\Delta}}+o(N^{3/2})\;,$$

where

$$\Delta \;=\; \frac{1}{18}\left[19-\frac{37}{(431-18\sqrt{417})^{1/3}}-(431-18\sqrt{417})^{1/3}\right]\;.$$

The near-horizon limit of the M2-branes at the Calabi-Yau singularity is now $\mathbf{AdS_4} \times \mathbf{Y_7}$ where $\mathbf{Y_7}$ is an Einstein manifold.

The Einstein metric is not known explicitly for this particular example, but due to work of Martelli-JFS-Yau we can compute its volume*, which is enough to determine the Euclidean action. [We also know it exists, due to Futaki-Ono-Wang in 2009.]

We precisely reproduce the above numerical factor from this geometrical calculation:

$$\mathrm{Vol}(\mathbf{Y_7}) \ = \ \frac{4-3\Delta}{96\Delta(1-\Delta)^2(2-\Delta)^2}\pi^4 \ .$$

[⋆ also using a fixed point theorem!]

- We have other interesting examples, *e.g.* a similar precise matching for a $\mathbf{U(1)} \times \mathbf{U(1)}$-invariant squashed $\mathbf{S^3}$.

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- The results I have presented are somewhat formal; more generally I can ask when can I put supersymmetric field theories on compact manifolds like this, and what does it mean physically?
- More importantly, these AdS/CFT computations always seem to work (when we can compute on both sides and compare, as in the examples I've discussed). Will we ever really understand why?!

Happy birthday Stephen!