Consistent Kaluza Klein Universes from Maximal Supergravity

The State of the Universe, 5-8 January 2012:
celebrating Stephen Hawking's 70th birthday

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Based on:

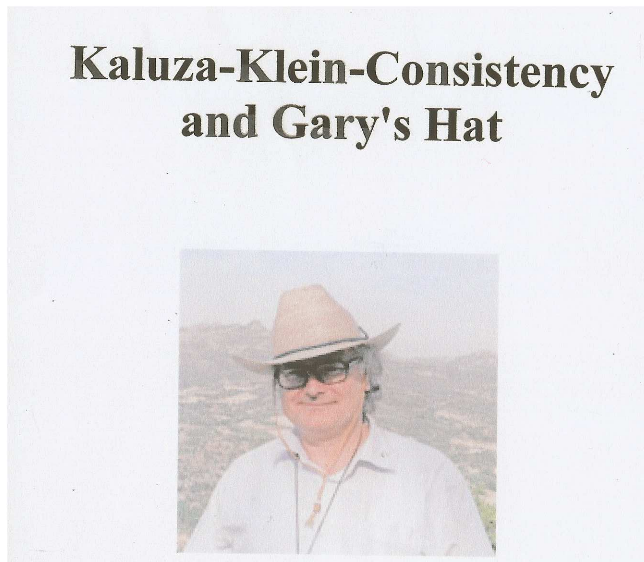
Krzysztof Pilch & HN, arXiv:1112.6131

B. de Wit & HN, NPB274(1986)363; NPB281(1987)211

The state of the universe on 25 July 1999

… relaxing after the Potsdam Strings ’99 Conference
(a smashing success thanks to Stephen!)

… continuing a previous birthday talk



Cambridge-Mitchell-Texas Conference, 20 - 26 August 2006

The Landscape of $N=8$ supergravity

70 scalars = local coordinates on $\mathbf{E}_{7(7)}/\mathbf{SU(8)}$ [Cremmer,Julia(1979)]

$$\mathcal{V}(x) = \begin{pmatrix} u_{ij}{}^{IJ} & v_{ijIJ} \\ v^{ijIJ} & u^{ij}{}_{IJ} \end{pmatrix} \in \mathbf{56} \text{ of } \mathrm{E}_{7(7)}$$

The potential of $SO(8)$ gauged $N=8$ supergravity is

$$\mathcal{P}(\mathcal{V}) = g^2 \left(-\frac{3}{4} \left| A_1^{ij} \right|^2 + \frac{1}{24} \left| A_{2i}{}^{jkl} \right|^2 \right)$$

$\rightarrow$ non-polynomial function on coset space $\mathbf{E}_{7(7)}/\mathbf{SU(8)}$ which is unbounded from below (and above).

A_1^{ij} and $A_{2i}{}^{jkl}$ from T tensor (in $\underline{912}$ irrep of $\mathbf{E}_{7(7)}$)

$$\begin{aligned} T_i^{jkl} &= \left(u^{kl}{}_{IJ} + v^{klIJ}\right)\left(u_{im}{}^{JK} u^{jm}{}_{KI} - v_{imJK} v^{jmKI}\right) \\ &= -\frac{3}{4} A_{2\,i}{}^{jkl} + \frac{3}{2} \delta_i^{[k} A_1^{l]j} \end{aligned}$$

[**Modern description via *embedding tensor formalism*:**

[HN,Samtleben (2001); deWit,Samtleben,Trigiante(2004)]]

Stationary points with $SU(3) \subset G_0 \subset SO(8)$

$$P = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix} \times \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} . \qquad (2.8)$$

Then an arbitrary element $y \in X$ may be written as

$$y = S(\lambda G_1^+ + \lambda' G_2^+) , \qquad (2.9)$$

where the action of S is that of SU(8) on its 70-dimensional representation in the space of self-dual complex four-forms. In the notation of ref. [1], $G_1^+ = X_1 + X_2 + X_3$ and $G_2^+ = X_4 + X_5 + X_6 + X_7$.

Using this parametrization of X, one readily obtains an explicit form for the potential

$$V = \tfrac{1}{8} g^2 \{ s'^4 [(x^2+3)c^3 + 4x^2 v^3 s^3 - 3v(x^2-1)s^3 + 12 x v^2 c s^2 - 6(x-1)cs^2 + 6(x+1)c^2 sv] + 2s'^2 [2(c^3 + v^3 s^3) + 3(x+1)us^3 + 6xv^2 cs^2 - 3(x-1)cs^2 - 6c] - 12c \} , \qquad (2.10)$$

where

$$c = \cosh(\lambda/\sqrt{2}) , \quad s = \sinh(\lambda/\sqrt{2}) ,$$
$$c' = \cosh(\lambda'/\sqrt{2}) , \quad s' = \sinh(\lambda'/\sqrt{2}) ,$$
$$v = \cos\alpha , \quad x = \cos 2\varphi . \qquad (2.11)$$

The fact that V is independent of ϑ reflects the SO(8) invariance of the potential. However, the fact that it is independent of ψ is a surprise, and possibly reflects a larger invariance of the SU(3) singlet sector.

It is now fairly straightforward, but tedious, to find the critical points of (2.10). It is important to check the singular points of the parametrization (2.9), since these conceal some true critical points of the potential [1].

Apart from the trivial extremum, where all the scalars vanish, there are five new solutions. These are listed in table 1 with their defining parameters, cosmological constants, symmetry groups, and remaining supersymmetry. The generic forms of the scalar fields at the extrema, listed in column 3, are obtained by substituting the parameter values of column 2 into (2.7), (2.8) and (2.9). The values of μ and μ' are not given, but may easily be obtained in terms of λ, λ',

Table 1
The extrema of the scalar potential for which the remaining symmetry is at least SU(3).

Extremum	Parameter values	Generic form of the scalar field at the extremum	Scalar pseudoscalar or mixture	Cosmological constant Λ	Remaining group symmetry	Remaining supersymmetry
E_1	$s = \pm 1/2$, $s' = 1/2$; $\alpha = \phi = 0$	$i\mu(G_1^- + G_2^-)$	P	$-25\sqrt{5}g^2/8$	$SO(7)^-$	None
E_2	$s = s' = [(3/\sqrt{5} - 1)/2]^{1/2}$; $\alpha = \phi = 0$	$\mu(G_1^+ + G_2^+)$	S	$-2 \times 5^{3/4} g^2$	$SO(7)^+$	None
E_3	$s = \pm[2(\sqrt{3} - 1)/5]^{1/2}$; $s' = [2(\sqrt{3} - 1)/5]^{1/2}$; $\phi = \alpha = \arccos[(3 - \sqrt{3})^{1/2}/2]$	$\mu(G_1^+ + G_2^+) + i\mu'(G_1^- + G_2^-)$	M	$-216\sqrt{2}\,3^{1/4} g^2/25\sqrt{5}$	G_2	$N = 1$
E_4	$s = 0$, $s' = 1$; $\phi = \pi/2$ a)	$i\mu G_2^-$	P	$-8g^2$	$SU(4)^-$	None
E_5	$s = 1/\sqrt{3}$, $s' = 1/\sqrt{2}$; $\alpha = 0$, $\phi = \pi/2$	$\mu G_1^+ + i\mu' G_2^-$	M	$-9\sqrt{3}g^2/2$	$SU(3) \times U(1)$	$N = 2$

a) Since $s = 0$, the variable α becomes degenerate, and thus no value is given.

N.P. Warner, Phys. Lett. 128B (1983) 169 and PhD-THESIS

A strange coincidence?

$SO(8) \to SU(3)\times U(1)$ **breaking and 'family color locking'**

$$
\begin{array}{lcll}
(u\,,\,c\,,\,t)_L & : & \mathbf{3}_c \times \bar{\mathbf{3}}_f \to \mathbf{8} \oplus \mathbf{1}\,, & Q = \frac{2}{3} - q \\
(\bar{u}\,,\,\bar{c}\,,\,\bar{t})_L & : & \bar{\mathbf{3}}_c \times \mathbf{3}_f \to \mathbf{8} \oplus \mathbf{1}\,, & Q = -\frac{2}{3} + q \\
(d\,,\,s\,,\,b)_L & : & \mathbf{3}_c \times \mathbf{3}_f \to \mathbf{6} \oplus \bar{\mathbf{3}}\,, & Q = -\frac{1}{3} + q \\
(\bar{d}\,,\,\bar{s}\,,\,\bar{b})_L & : & \bar{\mathbf{3}}_c \times \bar{\mathbf{3}}_f \to \bar{\mathbf{6}} \oplus \mathbf{3}\,, & Q = \frac{1}{3} - q \\
(e^-, \mu^-, \tau^-)_L & : & \mathbf{1}_c \times \mathbf{3}_f \to \mathbf{3}\,, & Q = -1 + q \\
(e^+, \mu^+, \tau^+)_L & : & \mathbf{1}_c \times \bar{\mathbf{3}}_f \to \bar{\mathbf{3}}\,, & Q = 1 - q \\
(\nu_e\,,\,\nu_\mu\,,\,\nu_\tau)_L & : & \mathbf{1}_c \times \bar{\mathbf{3}}_f \to \bar{\mathbf{3}}\,, & Q = -q \\
(\bar{\nu}_e\,,\,\bar{\nu}_\mu\,,\,\bar{\nu}_\tau)_L & : & \mathbf{1}_c \times \mathbf{3}_f \to \mathbf{3}\,, & Q = q
\end{array}
$$

Supergravity and Standard Model assignments agree if spurion charge is chosen as $q = \frac{1}{6}$ [Gell-Mann (1983)]

Realized at $SU(3)\times U(1)$ **stationary point!** [Warner,HN, NPB259(1985)412]

*[... just in case LHC keeps *not* finding new fundamental spin-$\frac{1}{2}$ fermions...]*

Potential progress, at last (after 25 years)

−6.00000	$SO(8)$	$\mathcal{N}=8$	$\checkmark$	−6.68740	$SO(7)$	–	–	−6.98771	$SO(7)$	–	–
−7.19158	G_2	–	$\checkmark$	−7.79423	$SU(3)\times U(1)$	$\mathcal{N}=2$	$\checkmark$	−8.00000	$SU(4)$	–	–
−8.47214	$SO(3)\times SO(2)^2$	–	–	−8.69597	$SO(3)\times SO(2)$	–	–	−8.80734	$SO(3)\times SO(3)$	–	–
−9.83995	$SO(3)\times SO(2)$	–	–	−9.98708	$SO(2)$	–	–	−10.06758	$SO(2)$	–	–
−10.39625	$SO(2)$	–	–	−10.43471	–	–	–	−10.46018	–	–	–
−10.67475	$SO(2)^2$	–	–	−10.68972	$SO(2)^2$	–	–	−10.75829	$SO(3)\times SO(2)$	–	–
−11.65685	$SO(2)^2$	–	–	−11.76725	–	–	–	−11.95898	$SO(3)$	–	–
−12.00000	$SO(2)^2$	$\mathcal{N}=1$	$\checkmark$	−12.12987	$SO(2)$	–	–	−12.71622	$SO(3)$	–	–
−13.01602	$SO(2)^2$	–	–	−13.62365	$SO(2)$	–	–	−13.63783	–	–	–
−13.66865	–	–	–	−13.67611	–	–	–	−13.79440	–	–	–
−13.84136	$SO(2)$	–	–	−14.00000	$SO(3)\times SO(3)$	–	$\checkmark$	−14.00056	–	–	–
−14.02217	$SO(2)$	–	–	−14.24026	$SO(3)$	–	–	−14.41574	$SO(2)$	–	–
−14.42019	$SO(2)$	–	–	−14.43835	$SO(2)$	–	–	−14.64498	–	–	–
−14.65354	–	–	–	−14.69694	$SO(2)$	–	–	−14.70986	–	–	–
−14.73608	–	–	–	−14.74271	–	–	–	−14.77609	–	–	–
−14.97038	$SO(2)$	–	–	−15.14243	–	–	–	−15.71628	$SO(2)$	–	–
−15.86254	–	–	–	−15.88533	–	–	–	−15.96185	–	–	–
−16.00000	–	–	–	−16.03495	–	–	–	−16.09268	–	–	–
−16.24005	–	–	–	−16.24010	–	–	–	−16.27388	–	–	–
−16.37793	–	–	–	−16.37803	–	–	–	−16.41446	–	–	–
−16.50531	–	–	–	−16.50772	–	–	–	−16.52032	–	–	–
−16.71973	$SO(2)$	–	–	−16.91171	–	–	–	−17.75879	–	–	–
−17.75886	–	–	–	−17.87646	–	–	–	−18.00000	–	–	–
−18.05269	–	–	–	−18.80177	$SO(2)$	–	–	−18.89269	$SO(2)$	–	–
−20.06989	$SO(2)$	–	–	−20.40657	–	–	–	−20.43648	–	–	–
−20.43966	–	–	–	−20.45402	$SO(2)$	–	–	−20.54313	–	–	–
−20.54715	$SO(2)$	–	–	−20.74862	–	–	–	−20.95412	–	–	–
−20.99419	–	–	–	−21.01265	–	–	–	−21.18474	–	–	–
−21.21742	–	–	–	−21.26598	–	–	–	−21.35328	–	–	–
−21.35978	–	–	–	−21.40849	–	–	–	−21.44362	–	–	–
−21.44750	–	–	–	−21.45935	–	–	–	−21.47656	–	–	–
−21.53575	–	–	–	−21.54972	–	–	–	−21.60232	–	–	–
−21.71188	–	–	–	−21.78669	–	–	–	−21.83496	–	–	–
−22.06714	–	–	–	−22.15487	$SO(2)$	–	–	−22.40837	–	–	–
−22.41557	–	–	–	−22.62631	–	–	–	−22.79258	$SO(2)$	–	–
−22.79860	$SO(2)$	–	–	−22.91945	$SO(2)$	–	–	−23.08861	–	–	–
−23.09630	–	–	–	−23.35620	–	–	–	−23.48986	–	–	–
−23.89434	$SO(2)$	–	–	−23.95246	–	–	–	−24.11233	–	–	–
−24.16856	–	–	–	−24.16941	–	–	–	−24.20275	–	–	–
−24.23138	–	–	–	−24.35198	$SO(2)$	–	–	−24.35908	–	–	–
−24.84339	–	–	–	−24.88182	–	–	–	−24.88242	–	–	–
−24.97037	–	–	–	−25.03105	$SO(3)$	–	–	−25.14937	–	–	–
−25.19962	$SO(2)^2$	–	–	−25.22870	$SO(2)$	–	–	−25.50397	–	–	–
−25.51137	–	–	–	−25.66527	–	–	–	−25.84003	–	–	–
−26.44795	–	–	–	−26.48510	–	–	–	−26.52206	–	–	–
−26.52377	–	–	–	−26.53754	–	–	–	−26.66808	–	–	–
−26.97506	$SO(2)$	–	–	−27.02581	–	–	–	−27.05606	–	–	–
−27.07529	$SO(2)$	–	–	−27.51936	$SO(2)$	–	–	−28.03141	–	–	–
−28.49862	–	–	–	−28.55585	$SO(2)$	–	–	−28.74576	–	–	–
−29.80210	$SO(2)$	–	–	−31.08383	$SO(2)$	–	–	−31.55402	–	–	–
−31.66154	–	–	–	−32.54262	–	–	–	−32.54576	–	–	–
−32.54929	–	–	–	−33.05154	–	–	–	−34.98681	–	–	–
−35.60884	–	–	–	−37.77270	–	–	–	−40.09883	–	–	–
−41.45121	–	–	–	−41.68086	–	–	–				

T. Fischbacher, "The encyclopedic reference of critical points for $SO(8)$ gauged $N=8$ supergravity" [arXiv:1109.1424[hep-th]]

Consistent KK reductions and maximal supergravity

Stationary points $\cong$ warped products $\mathbf{AdS}_4 \times \mathcal{M}^7$

$$ds^2_{11} = \Delta^{-1} ds^2_{\mathrm{AdS}_4} + ds^2_{\mathcal{M}^7} \quad , \qquad F = f\Delta^{-2}\mathrm{vol}_{\mathrm{AdS}_4} + \frac{1}{4!}F_{abcd}\, e^a \wedge e^b \wedge e^d \wedge e^d$$

$\Rightarrow$ a rich variety of new solutions for maximal supergravity via consistent embeddings? [Cf.Gibbons at DeserFest 2006]

Need: *Lift formulas* for metric and p-form fields!

- **$\mathbf{AdS}_4 \times \mathbf{S}^7$** [deWit & HN (1987); Pilch & HN (2011)]
- **$\mathbf{AdS}_7 \times \mathbf{S}^4$** [Nastase,Vaman,van Nieuwenhuizen (1999)]
- **$\mathbf{AdS}_5 \times \mathbf{S}^5$** **no complete proof as yet!**

But much work on consistent embeddings for non-maximal supergravities (with and without scalars)

[Cvetic,Lu,Pope; Gibbons; Gauntlett,Kim,Varela,Waldram;....]

Metric lift formula for SUGRA_{11}

'Generalized geometry' $\equiv$ **re-write** $D = 11$ **theory in terms of** $D\!=\!4$ **fields and symmetries (from 4+7 split).**

In particular: define new ('generalized') vielbein

$$e^m_{AB}(x,y) = i\Delta^{-1/2} e_a{}^m \big(\Phi^T \Gamma^a \Phi\big)_{AB}\,, \qquad \Delta \equiv \det\big(\mathring{e}_a{}^m e_m{}^b\big)$$

with 'alignment rotation' $\Phi(x,y) \in \mathrm{SU}(8)$**.**

Comparison with $D = 4$ **shows (with** $K^{m\,IJ} = \mathring{e}_a{}^m \eta^I \Gamma^a \eta^J$**)**

$$e^m_{ij}(x,y) \equiv e^m_{AB}\,\eta^A_i \eta^B_j = K^{m\,IJ}(y)\big(u_{ij}{}^{IJ}(x) + v_{ij\,IJ}(x)\big)$$

SU(8) invariant contraction yields lift formula [deWit,HN,Warner(1985)]

$$8\Delta^{-1} g^{mn}(x,y) = \big(K^{m\,IJ} K^{n\,KL}\big)(y)\big(u_{ij}{}^{IJ} + v_{ij\,IJ}\big)\big(u^{ij}{}_{KL} + v^{ij\,KL}\big)(x)$$

This formula has passed numerous non-trivial tests and can also be generalized to IIB on $\mathbf{AdS}_5 \times S^5$**.**

[Nilsson;Pope;Bobev,Gowdigere,Halmagyi,Pilch,Warner;Ahn,Itoh,Kundu;...]

Generalized Vielbein Postulate (GVP)

Generalized vielbein is *covariantly constant*

$$\mathring{D}_m e^n_{AB} + \mathcal{B}_m{}^C{}_{[A} e^n_{B]C} + \mathcal{A}_{m\,ABCD} e^{n\,CD} = 0$$

with respect to 'internal' $E_{7(7)}$ connection

$$\mathcal{B}_m{}^A{}_B = \frac{1}{2}(S^{-1}\mathring{D}_m S)_{ab}\Gamma^{ab}_{AB} + \frac{i\sqrt{2}}{14} f\, e_{ma}\Gamma^a_{AB} - \frac{\sqrt{2}}{48} e_m{}^a F_{abcd}\Gamma^{bcd}_{AB}$$

$$\mathcal{A}_{mABCD} = -\frac{3}{4}(S^{-1}\mathring{D}_m S)_{ab}\Gamma^a_{[AB}\Gamma^b_{CD]} + \frac{i\sqrt{2}}{56} e_{ma} f\, \Gamma^{ab}_{[AB}\Gamma^b_{CD]} + \frac{\sqrt{2}}{32} e_m{}^a F_{abcd}\Gamma^{[b}_{[AB}\Gamma^{cd]}_{CD]}$$

The flux components $F_{abcd}(x,y)$ and $24if(x,y) \equiv \varepsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta\gamma\delta}$ arise in the usual way from the 4-form field strength F_{ABCD} (here with *flat* indices). However, replacing

$$f e_{ma} \to e_m{}^b X_{b|a} \quad , \qquad e_m{}^a F_{abcd} \to e_m{}^a X_{a|bcd}$$

GVP still holds $\Rightarrow$ must ensure that flux lift formulas respect tensor structure as inherited from $D=11$!

GVP from $D = 4$

Replacing $e^m_{AB} \to e^m_{ij}$ **we have an analogous GVP**

$$\mathring{D}_m e^n_{ij} + \mathcal{B}_m{}^k{}_{[i}\, e^n_{j]k} + \mathcal{A}_{m\,ijkl}\, e^{n\,kl} = 0$$

With $\alpha + 4\beta = 1$ **this is solved by**

$$\begin{aligned}
\mathcal{B}_m{}^i{}_j(x,y) \;&=\; -\frac{2}{3}\alpha K^{IJ}_m \big(u^{ik}{}_{IK} u_{jk}{}^{JK} - v^{ik\,IK} v_{jk\,JK} \big) \\
&\qquad -\frac{2}{3}\beta \mathring{D}_m K^{[IJ}_n K^{n\,KL]} \big(v^{ik\,IJ} u_{jk}{}^{KL} - u^{ik}{}_{IJ} v_{jk\,KL} \big) \\
\mathcal{A}_{m\,ijkl}(x,y) \;&=\; \alpha K^{IJ}_m \big(v_{ij\,JK} u_{kl}{}^{IK} - u_{ij}{}^{JK} v_{kl\,IK} \big) \\
&\qquad -\beta \mathring{D}_m K^{[IJ}_n K^{n\,KL]} \big(u_{ij}{}^{IJ} u_{kl}{}^{KL} - v_{ij\,IJ} v_{kl\,KL} \big)
\end{aligned}$$

in terms of $D = 4$ **quantities and Killing vectors** K^{IJ}_m.

To eliminate remaining freedom we need extra information $\to$

The '$\mathfrak{A}$-equations'

First show that the $\mathfrak{A}_1$ and $\mathfrak{A}_2$ tensors

$$\mathfrak{A}_1^{ij} := -\frac{1}{4}\,(e^{m\,ik}\,\mathcal{B}_m{}^j{}_k + \mathcal{A}_m{}^{ijkl} e^m_{kl})\,,$$

$$\mathfrak{A}_{2l}{}^{ijk} := -\frac{1}{4}\,(3\,e^{m\,[ij}\mathcal{B}_m{}^{k]}{}_l - 3\,e^m_{pq}\,\mathcal{A}_m{}^{pq[ij}\delta^{k]}{}_l - 4\,\mathcal{A}_m{}^{ijkp} e^m_{pl})\,,$$

transform in 912 $\in$ 56 $\times$ 133 of $\mathbf{E}_{7(7)}$.

Detailed comparison of $D=11$ and $D=4$ supersymmetry variations then leads to identification with A_1 and A_2 tensors of $N=8$ supergravity:

$$\mathfrak{A}_1^{ij}(x,y) \;=\; A_1^{ij}(x)$$
$$\mathfrak{A}_{2\,l}{}^{ijk}(x,y) \;=\; A_{2\,l}{}^{ijk}(x)$$

Requiring y-independence of LHS fixes $\alpha = \frac{4}{7}\,,\;\beta = \frac{3}{28}$.
(... a long and rather tedious calculation...)

Standard Inhomogeneous Solution

$$\mathring{\mathcal{B}}_m{}^i{}_j = -\frac{8}{21} K_m^{IJ} \big(u^{ik}{}_{IK} u_{jk}{}^{JK} - v^{ik\,IK} v_{jk\,JK} \big) - \frac{1}{14} \mathring{D}_m K_n^{[IJ} K^{n\,KL]} \big(v^{ik\,IJ} u_{jk}{}^{KL} - u^{ik}{}_{IJ} v_{jk\,KL} \big)$$

$$\mathring{\mathcal{A}}_{m\,ijkl} = \frac{4}{7} K_m^{IJ} \big(v_{ij\,JK} u_{kl}{}^{IK} - u_{ij}{}^{JK} v_{kl\,IK} \big) - \frac{3}{28} \mathring{D}_m K_n^{[IJ} K^{n\,KL]} \big(u_{ij}{}^{IJ} u_{kl}{}^{KL} - v_{ij\,IJ} v_{kl\,KL} \big)$$

$\Rightarrow$ **complete non-linear $D\!=\!4$ SUSY *etc.* from KK**

However, when rotated back into $D = 11$ expressions required tensor structure is *not* satisfied in general.

Resolution: **must add a homogeneous solution to GVP which is 'in the kernel' of the $\mathfrak{A}$-equations!** [Pilch & HN (2011)]

$\Leftrightarrow$ **there exists modification $\mathring{\mathcal{B}} \rightarrow \mathring{\mathcal{B}} + \delta\mathcal{B}$ and $\mathring{\mathcal{A}} \rightarrow \mathring{\mathcal{A}} + \delta\mathcal{A}$ such that fluxes acquire correct tensor structure.**

The non-linear flux formulas

Idea: **flux lift formulas by comparison** $D=4$ ***vs.*** $D=11$**.**

Complication: **SU(8) rotation** $\Phi(x,y)$ **is non-trivial and not generally known (or constructible) in closed form.**

Resolution: **SU(8)-invariant contractions** $\rightarrow$

$$f(x,y) \;=\; -\frac{\sqrt{2}}{48\cdot 5!}\,\Delta^4\, g^{mm'}\,\varepsilon_{mnpqrst}\, e^n_{ij}\,(e^{[p}\bar{e}^q e^r\bar{e}^s e^{t]})_{kl}\,\mathcal{A}_{m'}{}^{ijkl}$$

$$F_{mnpq}(x,y) \;=\; -\frac{i}{144}\,\Delta^4\, g_{rw}\, e^r_{ij}(e^{[s}\bar{e}^t e^u\bar{e}^v e^{w]})_{kl}\,\varepsilon_{stuv[mnp}\,\mathcal{A}_{q]}{}^{ijkl}$$

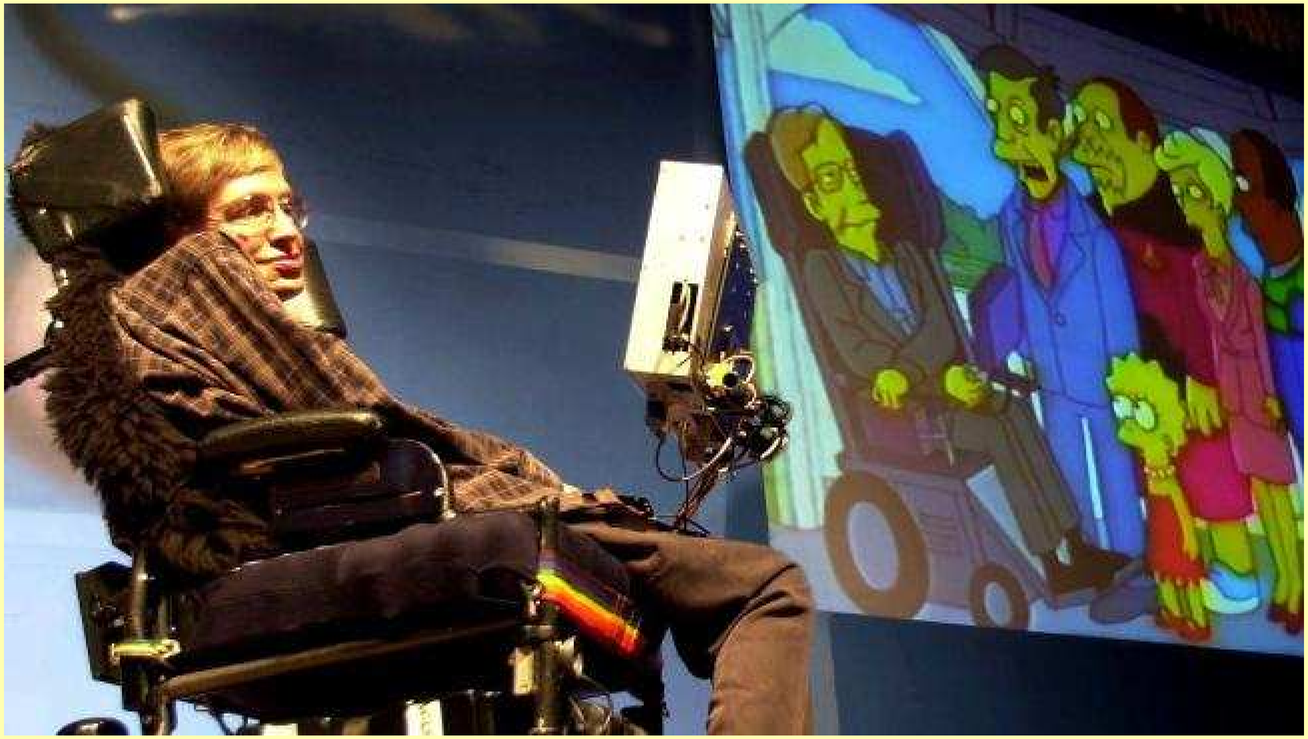
RHS entirely expressible in terms of $D=4$ **quantities (but must first work out non-linear metric** $g_{mn}(x,y)$**).**

Remarkable fact: **fluxes are** ***invariants*** **of GVP and** $\mathfrak{A}$**-equations** $\rightarrow$ **can also use** $\mathring{\mathcal{A}}_{m\,ijkl}$ **to evaluate them!**

Outlook

- **Flux lift formulas only valid *on-shell* in general.**
- **Flux lift formulas have been tested analytically and numerically for several non-trivial stationary points.**
- **Rich potential landscape $\Rightarrow$ many new $D=11$ solutions from consistent embeddings**
- **Even if we are not yet sure about the state of the universe: if it is part of a higher-dimensional universe it better be a consistent embedding!**

Happy Birthday, Stephen!



[from http://resources2.news.com.au/...]

... and many more universes!