# Conformal Symmetry and Scaling Limits of Black Holes

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There has been a great deal of work since the late 1990's up to to the present day on on the conformal symmetries of (non-extreme) black holes. Much of it by people at this conference.

The aim of this talk, which is an outcome * of a visit to Philadelphia last term, is to review and "deconstruct" a series of papers on this subject written over the years by Mirjam Cvetic and Finn Larsen

*M. Cvetic and G. W. Gibbons, Conformal Symmetry of a Black Hole as a Scaling Limit: A Black Hole in an Asymptotically Conical Box," arXiv:1201.0601 [hep-th].

**Plan of the talk**

•Previous Work•

•Recent Developments•

•Scaling Limits•

•Harrison Transformations•

•Asymptotics•

•Symmetries•

•Previous Work: It has been known since the early 1980's that black holes in (ungauged) supergravity theories (now thought of as solitons of string theory) have many remarkable properties and that in particular one may use solution generating techniques based on symmetries of the equations of motion now known as S and T dualities to generate charged rotating solutions starting from the standard Kerr Solution.

The most general known explicit solution in four spacetime dimensions was obtained by Cvetic and Youm in 1996 and in a different form (and in gauged supergravity) by Chong,Cvetic, Lu and Pope in 2006,

$\mathcal{N} = 2$ SUGRA coupled to three vector multiplets.

$$\begin{aligned}
\mathcal{L}_4 &= R*\mathbf{1} - \frac{1}{2}*d\varphi_i \wedge d\varphi_i - \frac{1}{2}e^{2\varphi_i}*d\chi_i \wedge d\chi_i - \frac{1}{2}e^{-\varphi_1}(e^{\varphi_2-\varphi_3}*F_1 \wedge F_1 \\
&+ e^{\varphi_2+\varphi_3}*F_2 \wedge F_2 + e^{-\varphi_2+\varphi_3}*\mathcal{F}_1 \wedge \mathcal{F}_1 + e^{-\varphi_2-\varphi_3}*\mathcal{F}_2 \wedge \mathcal{F}_2) \\
&- \chi_1(F_1 \wedge \mathcal{F}_1 + F_2 \wedge \mathcal{F}_2),
\end{aligned}$$

$$\begin{aligned}
F_1 &= dA_1 - \chi_2\, d\mathcal{A}_2, \\
F_2 &= dA_2 + \chi_2\, d\mathcal{A}_1 - \chi_3\, dA_1 + \chi_2\,\chi_3\, d\mathcal{A}_2, \\
\mathcal{F}_1 &= d\mathcal{A}_1 + \chi_3\, d\mathcal{A}_2, \\
\mathcal{F}_2 &= d\mathcal{A}_2.
\end{aligned}$$

The four-dimensional theory can be obtained from six-dimensions, by reducing the bosonic string action

$$\mathcal{L}_6 = R*\mathbf{1} - \frac{1}{2}e^{-\sqrt{2}\phi}*F_{(3)} \wedge F_{(3)}$$

$$ds_4^2 = -\Delta_0^{-1/2} G (dt + \mathcal{A})^2 + \Delta_0^{1/2} \left( \frac{dr^2}{X} + d\theta^2 + \frac{X}{G} \sin^2\theta d\phi^2 \right) ,$$

$$\begin{aligned} X &= r^2 - 2mr + a^2 , \\ G &= r^2 - 2mr + a^2 \cos^2\theta , \\ \mathcal{A} &= \frac{2ma \sin^2\theta}{G} \left[ (\Pi_c - \Pi_s) r + 2m\Pi_s \right] d\phi , \end{aligned}$$

$$\Delta_0 = \prod_{I=1}^{4} (r + 2m \sinh^2\delta_I) + 2a^2 \cos^2\theta [r^2 + mr \sum_{I=1}^{4} \sinh^2\delta_I + 4m^2(\Pi_c - \Pi_s$$
$$-2m^2 \sum_{I<J<K} \sinh^2\delta_I \sinh^2\delta_J \sinh^2\delta_K] + a^4 \cos^4\theta .$$

$$\Pi_c \equiv \prod_{I=1}^{4} \cosh\delta_I , \quad \Pi_s \equiv \prod_{I=1}^{4} \sinh\delta_I .$$

In 1997 its was shown by by Cvetic and Larsen, that the massless scalar wave equation separates in these backgrounds just as it does in the Kerr-Solution. A partial explanation for this remarkable result can be found in recent work on Conformal-Killing Yano tensors by Page, Frolov, Kubiznak, Yasui and Houri, however it remains somewhat of a mystery.

Unfortunately, the resulting radial equations (and angular in the rotating case) have two regular and one irregular singular point, which impedes further analytic progress.

In particular, the radial equation has two regular singular points and a confluent singularity at infinity familiar from elementary quantum mechanical scattering theory.

Many authors have had the idea of mutilating the equations by hand so as to convert the irregular singular point at infinity to a regular singular point, thus allowing exact solutions in terms of hypergeometric functions.

This is has been especially attractive to string theorists since the hypergeometric equation is widely believed to have something to do with "conformal symmetry". What is true is that representations of $SL(2,\mathbb{R})$ can be made up of solutions of the hypergeometric equation.

On way to see this is to recall that the bi-invariant metric on $SL(2,\mathbb{R})$ coincides with the standard metric on $AdS_3$ which is therefore $(SL_L(2,\mathbb{R}) \times SL_R(2,\mathbb{R})/\mathbb{Z}_2) \equiv SO(2,2)$ invariant. The generators of $L_i$ and $R_i$ give 6 vector fields and acting on functions $f$ on $SL(2,\mathbb{R})$ aka $AdS_3$

$$\boxed{\nabla^2 f = L_i L_i f = R_i R_i f}$$

However, although it is true that $SO(2,2)/\mathbb{Z}_2$ is the conformal group of compactified 2-dimensional Minkowski spacetime $\mathbb{E}^{1,1}$ or indeed $AdS_2$ or $dS_2$ the relevance of this fact is not obvious.

Since $L_3$ and $R_3$ commute, the massive wave equation on $AdS_3$, $-L_iL_i = m^2$ admits separation of variables: $f = e^{i(\sigma s - \nu t)}f(r, \nu, \sigma)$.

If we introducing two Killing coordinates $s, t$ such that $L_3 = \partial_s - \partial_t$, $L_3 = \partial_s + \partial_t$ we may we may convert $L_i, R_i$ to 6 purely radial operators acting on $f(r, \nu, \sigma)$ with the same algebra and same relation between $\nabla^2$ and the Casimir we showed earlier

In 1997 Cvetic and Larsen found that if they passed from the exact geometry of the Cvetic-Youm black holes to a subtracted geometry designed to

- Maintain Separability

- Convert the radial equation to the Hypergeometric Equation

- Introduce an apparent $SO(3)$ symmetry

They found that the massless wave equation could be cast in the form

$$\boxed{\check{R}_i\check{R}_i = \check{L}_i\check{L}_i = l(l+1) = -\check{J}_i\check{J}_i}$$

where $\check{R}_i$ and $\check{L}_i$ are radial operators with the Lie algebra of $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ and $l(l+1)$ should be thought of as the Casimir $\check{J}_i\check{J}_i$ $SO(3)$, where $\check{J}_i$ are vector fields acting on $S^2$ with Lie brackets of $so(3)$.

Mysteriously, the complicated spheroidal Laplacian has turned into the much simpler Laplacian on the unit round 2-sphere $S^2$ and we seem to have acquired an $sl(2,\mathbb{R})\oplus sl(2,\mathbb{R})\times so(3)$ algebraic structure of some form.

The subtraction process consists in the replacements

$$\Delta_0 \rightarrow (2m)^3 r(\Pi_c^2 - \Pi_s^2) - (2m)^2(\Pi_c - \Pi_s)^2 a^2 \cos^2\theta \,.$$
$$\mathcal{A}_\phi \rightarrow \frac{2ma\sin^2\theta}{G}\Big[(\Pi_c - \Pi_s)r = 2m\Pi_s\Big]\,,$$

Reducing the highest power of $r$ in $\Delta_0$ renders the irregular singular point at infinity regular.

Note: the separation properties of the massless wave equation hold for any frequency $\omega$ and angular momentum $m$ of the massless wave equation and the horizon is in general non-extreme.

Last summer Cvetic and Larsen discovered further amazing properties of these metrics, among which are

- The subtracted metrics may be lifted to five spacetime dimensions where they take the form of the metric product $AdS_3 \times 4S^3$

- They thus solve the equations of simple supergravity in 4+1 spacetime dimensions

- They thus solve the equations of supergravity in 3+1 spacetime dimensions of the form given above plus a gravi-scalar and a gravi-photon.

• Recent Developments We have

• Obtained the subtracted metrics as a suitable scaling limit of the unsubtracted metrics

• Obtained the subtracted metrics as a suitable "infinite boost" Harrison transformation of the unsubtracted metrics

• Understood the asymptotics of the metric as being Asymptotically Conical, (AC)

• Seen more clearly in what sense $(SL(2\mathbb{R}) \times SL(2,\mathbb{R})/\mathbb{Z}_2) \times SO(3)$ is , and is not, a symmetry, and how its algebra acts on the four dimensional spacetime.

• Scaling Limits: In the static case, without loss of generality we take three equal charges and the fourth one different by defining $*F_1 = F_2 = *\mathcal{F}_1 \equiv F$ and $\mathcal{F}_2 \equiv \mathcal{F}$. * We use the "tilde" notation for all the variables, with the choice of charge parameters $\tilde{\delta}_1 = \tilde{\delta}_2 = \tilde{\delta}_3 \equiv \tilde{\delta}$ and $\tilde{\delta}_4 \equiv \tilde{\delta}_0$.

We take the following scaling limit with $\epsilon \to 0$:

$$\tilde{r} = r\epsilon, \quad \tilde{t} = t\epsilon^{-1}, \quad \tilde{m} = m\epsilon\,,$$
$$2\tilde{m}\sinh^2\tilde{\delta} \equiv Q = 2m\epsilon^{-1/3}(\Pi_c^2 - \Pi_s^2)^{1/3}, \quad \sinh^2\tilde{\delta}_0 = \frac{\Pi_s^2}{\Pi_c^2 - \Pi_s^2}\,,$$

*While one can in principle perform a scaling limit with three unequal large charges $Q_i$ $(I = 1, 2, 3)$, by replacing in the scaling limit $Q \to (\Pi_{I=1}^3 Q_I)^{\frac{1}{3}}$, appropriate powers of $Q_I$ in the scalar fields $\varphi_i$ $(i = 1, 2, 3)$ and gauge field strengths $*F_1, F_2, *\mathcal{F}_1$ can be removed without loss of generality, resulting in the same gauge choice for sources .

• Harrison Transformations: Consider static solutions to Einstein-Dilaton-Maxwell equations with the general dilation coupling $\alpha$.

$$
\begin{aligned}
\mathcal{L}_4 &= \sqrt{-g}\Big(\frac{1}{4}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-2\alpha\phi}F^2\Big)\,.\\
ds^2 &= -e^{2U}dt^2 + e^{-2U}\gamma_{ij}dx^i dx^j\,, \qquad F_{i0} = \partial_i\psi\\
\mathcal{L}_3 &= \sqrt{\gamma}\Big(R(\gamma_{ij}) - 2\gamma^{ij}\Big(\partial_i U\partial_j U + \partial_i\phi\partial_j\phi - e^{-2U}e^{-2\alpha\phi}\partial_i\psi\partial_j\psi\Big)\Big)
\end{aligned}
$$

$$x \equiv \frac{U+\alpha\phi}{\sqrt{1+\alpha^2}}\,, \qquad y \equiv \frac{-\alpha U+\phi}{\sqrt{1+\alpha^2}}\,,$$

$$\begin{aligned}
P &= e^{-\sqrt{1+\alpha^2}(x+y)} \begin{pmatrix} e^{2\sqrt{1+\alpha^2}x} - (1+\alpha^2)\psi^2 & -\sqrt{1+\alpha^2}\psi \\ -\sqrt{1+\alpha^2}\psi & -1 \end{pmatrix}\,, \\
\mathcal{L}_3 &= \sqrt{\gamma}\Big( R(\gamma_{ij}) + \frac{1}{1+\alpha^2}\gamma^{ij}\mathrm{Tr}(\partial_i P \partial_j P^{-1})\Big)\,, \\
P &\to HPH^T\,, \qquad H = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \in SO(1,1) \in SL(2,\mathbb{R})
\end{aligned}$$

We set $\alpha^2 = \frac{1}{3}$ and $H = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. In fact an infinite boost of Schwarzschild gives Robinson Bertotti.

- Asymptotics: The metrics are asymptotically conical, AC

$$\boxed{ds^2 \approx \Big(-\frac{R}{R_0}\Big)^6 dt^2 + 16dR^2 + R^2\Big(d\theta^2 + \sin^2\theta^2 d\phi^2)\Big)}$$

with deficit angle $\frac{\pi}{2}$ Since $|g_{tt}|$ increases monotonically without bound in the radial direction our black holes are confined within an AC box. If $\mathbf{x} = R^4(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$\boxed{ds^2 \approx \frac{1}{|\mathbf{x}|^{\frac{3}{2}}}\Big\{ - \Big(R_0^{-6}|\mathbf{x}|^3 dt^2 + |d\mathbf{x}|^2\Big\}\Big)}$$

$$\begin{aligned} \mathbf{x} &\rightarrow \lambda\mathbf{x} \qquad t \rightarrow \lambda^{-\frac{1}{2}} t \\ ds^2 &\rightarrow \lambda^{\frac{1}{2}} ds^2 \,. \end{aligned}$$

The Lifshitz scaling exponent $z = -\frac{1}{2}$.

The metrics are asymptotically conical because the energy density falls of as $\frac{1}{R^2}$ and the ADM energy is infinite. The same phenomenon was found by Clive Wells and myself in some flat space monopole solutions of Maxwell-Dilaton theory. The dilaton $\phi$ in these theories provides a spacetime dependent abelian gauge coupling constant $g$ which runs with $r$ . Equivalently one has spacetime dependent magnetic permeability and electric permittivity .

$$\boxed{g = e^{\alpha\phi}, \qquad \mu = g^2, \qquad \epsilon = g^{-2}}$$

$$D_r = E_r e^{-2\alpha\phi} = \frac{Q}{r^2}\,, \qquad B_r = \frac{P}{r^2}\,,$$

$$\begin{aligned}
&& \phi &= A \ln r + B \\
\text{If} \quad P = 0 \quad \text{then} && A &= \frac{1}{\alpha} = \alpha Q^2 e^{2\alpha B}\,, \qquad g^2 \propto r^2 \\
\text{If} \quad P = 0 \quad \text{then} && A &= -\frac{1}{\alpha} = -\alpha P^2 e^{-2\alpha B}\,, \qquad g^2 \propto \frac{1}{r^2} \\
&& T_{00} &= \propto \frac{1}{\alpha^2 r^2}\,.
\end{aligned}$$

• Symmetries From the four-dimensional point of view, what is most puzzling is the fact that the subtracted metrics are of co-homogeneity 2 :they admit only $\partial_t$ and $\partial_\phi$ as Killing fields and yet there appears to be some sort of action of $(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))/\mathbb{Z}_2 \times SO(3)$. There are certainly nine vector fields which $\check{R}_i, \check{L}_i, \check{J}_i$ which act on the four-dimensional subtracted spacetime and whose Lie-Brackets are those of $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(3)$ and which commute with the massless wave equation.

The explanation is that we have quotient of $AdS_3 \times 4S^2$, whose isometry group is generated by the 6 vectors fields $R_i, L_i, \check{J}_i$ by a one parameter subgroup of $(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))/\mathbb{Z}_2 \times SO(3)$ generated by a Killing vector $K$ field which has, in general has a projection on all three summands of $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(3)$ . If the projection on $o(3)$ vanishes we have a product of the BTZ black hole with $S^2$.

Now generically any element $K$ in $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(3)$ commutes with at most three other elements of $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus so(3)$ including itself. The two, $\partial_t$ and $\partial_\phi$, complementary to $K$, i.e its centraliser, descends to the quotient. The remaining generators are "broken".

$$
\begin{aligned}
ds_5^2 &= -\frac{X}{\rho}dt^2 + \frac{dr^2}{X} + \rho(d\alpha + \frac{[(\Pi_c - \Pi_s)r + 2m\Pi_s]}{(\Pi_c - \Pi_s)\rho}dt)^2 \\
&+ d\theta^2 + \sin^2\theta(d\phi + 2ma(\Pi_c - \Pi_s)d\alpha)^2. \\
\rho &= 8m^3[r(\Pi_c^2 - \Pi_s^2) + 2m\Pi_s^2 - \frac{a^2}{2m}(\Pi_c - \Pi_s)^2]\,.
\end{aligned}
$$

where $K = \frac{\partial}{\partial\alpha}$. Obviously, not all of $L_i$ and $R_i$ can commute with $K = \frac{\partial}{\partial\alpha}$ . To obtain $\check{R}_i$ and $\check{L}_i$ from $R_i$ and $L_i$ we need their action on functions defined on the four-dimensional subtracted geometry. Such functions may be pulled back to $AdS_3 \times 4S^2$ and satisfy

$$
\frac{\partial}{\partial\alpha}f = 0.
$$

Put simply, we substitute $\frac{\partial}{\partial\alpha}$ in the expressions for $L_i$ and $R_i$.

One may introduce formal operators on the subtracted geometries

$$\mathcal{L}_i = \frac{1}{\sqrt{-1}}\check{L}_i\,, \qquad \mathcal{R}_i = \frac{1}{\sqrt{-1}}\check{R}_i\,,$$
$$\mathcal{L}_\pm = \mathcal{L}_1 \pm i\mathcal{L}_2\,, \qquad \mathcal{R}_1 \pm i\mathcal{R}_2$$

And one has

$$\begin{aligned}\mathcal{R}_3 &= \frac{\beta_R}{\pi}i\partial_t + \frac{\beta_H\Omega_H}{\pi}i\partial_\phi = \frac{\beta_R}{\pi}\omega - \frac{\beta_H\Omega_H}{\pi}m\,,\\ \mathcal{L}_3 &= \frac{\beta_L}{\pi}i\partial_t = \frac{\beta_L}{\pi}\omega\end{aligned}$$

If we start from a state $|\psi\rangle$ for which $\mathcal{L}_3|\psi\rangle = 0$ then states of the form $(\mathcal{L}_+)^{n_L}|\psi\rangle$, $n_L \in \mathbb{N}$ will have

$$\boxed{\omega = \frac{2\pi}{\beta_L} n_L \,,}$$

and if they are of the form $(\mathcal{L}_+)^{n_R}|\psi\rangle$, $n_R \in \mathbb{N}$ they will have

$$\boxed{\frac{\beta_R}{2\pi}\omega - \frac{\beta_H \omega_H}{2\pi} m = n_R \,.}$$

A remaining puzzle is what is the inner product on these states? We find (formally)

$$\langle \phi_1 | \phi_2 \rangle = \int (\bar{\phi}_1 \phi_2) d\rho dt d\phi \sin\theta d\theta \,.$$

which is neither the Spacetime $L^2$ norm nor the Klein Gordon Inner product.

This talk has been about the $3+1$ -dimensional theory, because that will probably be of greatest interest of most of the audience. In particular our results apply to the subtracted geometries of Kerr-Newman black holes and may provide useful analytic approximations for the near horizon geometry of Non-extreme Astrophysical Black Holes which is different from the Bardeen-Horowitz NEHK geometry.

In our paper we also treat three-charged rotating black holes in $4+1$ spacetime dimensions. Everything goes through as in $3+1$ dimensions with some minor changes such as

$$AdS_3 \times 4S^2 \rightarrow AdS_3 \times S^3$$

and the Hopf fibration comes into play.

## Some Future Directions

•Thermodynamics and the notion of mass for AC metrics•

•Relation of algebra to Quantum Field Theory: nature of inner product•