G. W. Gibbons

The gravitational memory effect: what it is and why Stephen and I did not discover it

Gravity and Black Holes
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5th July 2017
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This talk is based on part of my thesis work carried out with Stephen from 1969-1972, and work over the years with Christian Duval, Peter Horvathy and Pengming Zhang , much of it carried out at LMTP in Tours supported by a LE STUDIUM chair under the aegis of a collaborative project entitled: Classical and Quantum Space-Time and its Symmetries An important contribution was a remark by Shahar Hadar over coffee.

The talk falls into three parts.

- A recollection of my first paper written with Stephen on the detection of gravitional waves useing bar detectors.
- A brief introduction and overview of the Carroll Group
- An application to plane gravitational waves, gravitational memory and its relation to notion of a soft graviton.

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[*] P. M. Zhang, C. Duval, G. W. Gibbons and P. A. Horvathy, The Memory Effect for Plane Gravitational Waves, `arXiv:1704.05997 [gr-qc]`.`version 2 out this morming`

From:

G. W. Gibbons and S. W. Hawking, Theory of the detection of short bursts of gravitational radiation, Phys. Rev. D **4** (1971) 219

we take the following equations :

$$\frac{d^2x}{dt^2} + \frac{\omega_0}{Q}\frac{dx}{dt} + \omega_0^2 x = -c^2 l R_{1010}$$

$$R_{i0j0} = \frac{G}{3r}\frac{d^4 D_{ij}}{dt^4}(t-r)\,,$$

We deduced that if the quadrupole moment is initially and finally time independent, as might be expected for the gravitational collapse of a massive star, then three integrals of the signal must vanish

$$\int_{t_i}^{t_f} dt \int_{t_i}^{t} dt' \int_{t_i}^{t'} dt'' R_{0i0j}(t'') = 0,$$

In which case the signal must change sign at least three times. We provided a sketch of a signal which changed sign exactly three times which appears to have mislead some people to think that we had claimed that it must always change sign three times. By contrast for what is now called a *flyby* we pointed out that only

$$\int_{t_i}^{t_f} dt R_{0i0j}(t)$$

need vanish. We did not labour the point of how this might affect the displacement $x(t)$ of the detector after a pulse like signal has passed.

Later Zeldovich and Polnarev [*] were considering likeley signals from dense clusters of massive stars or collapsed objects who noted that that after a pulse has passed,according to linear theory the metric perturbation h_{ij} satisfies

$$\frac{d^2 h_{ij}}{dt^2} = 0\,.$$

whose solution is

$$h_{ij} = h^1_{ij} t + h^0_{ij}\,, \qquad h^1_{ij}, h^0_{ij} \quad \text{constant}$$

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[*] Ya. B. Zel'dovich and A. G. Polnarev, "Radiation of gravitational waves by a cluster of superdense stars," Astron. Zh. **51**, 30 (1974) [Sov. Astron. **18** 17 (1974)].

and stated that:

> *. . . another, nonresonance, type of detector is possible, consisting of two noninteracting bodies (such as satellites). the values of* h_{ij} *after the encounter of two objects differs fromthe value before the encounter. As a result the distance between a pair of free bodies should change, and in principle this effect might possibly serve as a nonresonance detector. [. . .] One should note that although the distance between the free bodies will change, their relative velocity will actually become vanishingly small as the flyby event concludes.*


Я.Б. ЗЕЛЬДОВИЧ
1914 – 1987
РОССИЯ RUSSIA-2014
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Subsequently * Braginsky and Grischuk dubbed this the Memory effect

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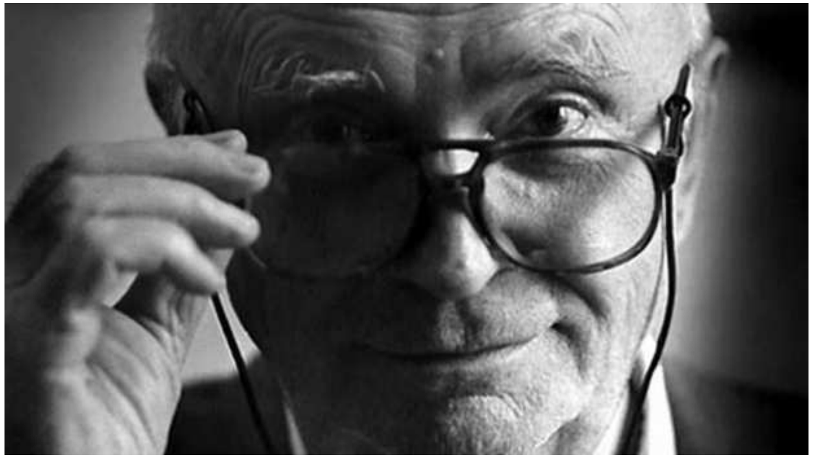
[*] V B Braginsky and L P Grishchuk, Kinematic resonance and the memory effect in free mass gravitational antennas, Zh. Eksp. Teor. Fiz. **89** 744-750 (1985) [Sov. Phys. JETP 62, 427 (1985)].

> *Consideration is given to two effects in the motion of free masses subjected to gravitational waves, kinematic resonance and the memory effect. In kinematic resonance, a systematic variation in the distance between the free masses occurs, provided the masses are free in a suitable phase of the gravitational wave. In the memory effect, the distance between a pair of bodies is different from the initial distance in the presence of a gravitational radiation pulse. Some possible applications [. . .] to detect gravitational radiation . . .*

Actually, as we have seen the distance can be expected to be time dependent in general.



The basic idea of our work is to look at non-Einsteinian Relativity Pinciples from an, albeit anachronistic, Spacetime view point

In our context a Principle of Relativity involves a notion of the invariance of physical laws under passing to a moving frame which we interpret as a symmetry of some sort of spacetime structure.

We follow the path pioneered by Bacry and Levy-Leblond [*] who found all algebras containing rotations, spatial and temporal translations and boosts. All may be regarded as Wigner-Inönü contractions [†] of the two De-Sitter groups.

Without boosts we would simply be classifying Aristotelian spacetimes which leads to Helmholtz's classification of congruence geometries [‡]
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[*] H. Bacry and J. Levy-Leblond, Possible kinematics J. Math. Phys. **9** (1968) 1605.

[†] E. Inönü , E.P. Wigner (1953). "On the Contraction of Groups and Their Representations". Proc. Nat. Acad. Sci. 39 (6): 51024.

[‡] Über die Thatsachen, welche der Geometrie zu Grunde liegen, in Wissenschaftliche Abhandlungen, Volume II, Leipzig: Johann Ambrosius Barth, 618639. Originally published in the Nachrichten von der Knigl. Gesellschaft der Wissenschaften zu Gttingen, No. 9 (3 June 1868).

The contractions are:

- Newton-Hooke $\Lambda \to O\,, \qquad c \to \infty\,, \qquad c^2\frac{\Lambda}{3}$ finite
- Poincaré $\Lambda \to O\,, \qquad c$ finite.
- Galilei $\Lambda \to O\,, \qquad c \to \infty.$
- Carroll $\Lambda \to O\,, \qquad c \to 0$

There is a certain duality between the Galilei and Carroll groups. In one the future light cone $t > \frac{1}{c}|\mathbf{x}|$ expands to become a future half space $t > 0$. In the other it contracts to become a future half line $t > 0\,, \mathbf{x} = 0$. One allows instantaneous propagation, the other is ultra-local and forbids any propagation.

All kinematic groups have a flat invariant model space time which allows a curved generalisation.

For Galilei this is Newton-Cartan spacetime with its degenerate co-metric g^{ij} whose kernel are co-normals of the absolute time slices

Carrollian spacetime. has a degenerate metric g_{ij} whose kernel is tangent to the absolute future [*].

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[*] To quote Mrs Thatcher: TINA, i.e. There is no alternative

Well, in our country,” said Alice, still panting a little, ”you’d generally get to somewhere else if you run very fast for a long time, as we’ve been doing.”

A slow sort of country!” said the Queen. ”Now, here, you see, it takes all the running you can do, to keep in the same place. If you want to get somewhere else, you must run at least twice as fast as that!”

Galilei, boosts act as

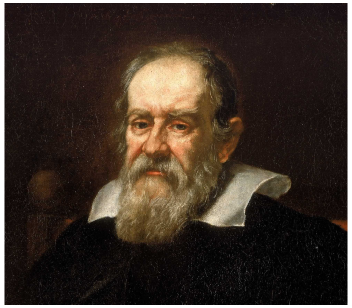
$$(t, \mathbf{x}) \to (t, \mathbf{x} - \mathbf{v}t)$$

Carroll, boosts act as

$$(s, \mathbf{x}) \to (s - \mathbf{b} \cdot \mathbf{x}, \mathbf{x})$$

where t is Galilean time and s is Carrollian time.

In 1+1 spacetime dimensions, Galileo and Carroll coincide as groups since we may interchange Galilean space and with Carrollian time and *vice versa*



Taking the limit $c \uparrow \infty$ in the contra-variant Minkowski co- metric

$$-\frac{1}{c^2}\frac{\partial}{\partial t} \otimes \frac{\partial}{\partial t} + \delta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}$$

motivates the definition of a Newton-Cartan Spacetime as a quadruple $\{N, \gamma, \theta, \nabla\}$ where N is a smooth $d+1$ manifold , γ a symmetric semi-positive definite contravariant 2-tensor of rank d with kernel the one-form θ and ∇ a symmetric affine connection w.r.t. which γ and θ are parallel.

Taking the limit $c \downarrow 0$ in the co-variant Minkowski metric

$$-c^2 dt^2 + \delta_{ij} dx^i dx^j$$

motivates the definition of a Carrollian Spacetime as a quadruple $\{C, g, \xi, \nabla\}$ where N is a smooth $d+1$ manifold , g a symmetric semi-positive definite co-variant 2-tensor of rank d with kernel the vector field ξ and ∇ a symmetric affine connection w.r.t. which ξ and ∇ are parallel.

The standard flat case is $C = \mathbb{R} \times \mathbb{R}^d$, $g_{ij} = \delta_{ij}$, $\xi = \frac{\partial}{\partial s}$, $\Gamma_\mu{}^\nu{}_\lambda = 0$ where s is Carrollian time. The isometry group of the Carrollian metric contains

$$x^i \to x^i\,, \qquad s \to s + f(x^i)$$

and so is infinite dimensional but if we require that the Carrolian automorphisms preserve the connection ∇ we obtain the standard finite dimensional Carroll group.

All the kinematic groups have a description in terms of Lorentzian geometry in 4+1 spacetime dimensions.

- Minkowski spacetime arises from a Kaluza-Klein reduction on a spacelike translation as shown by Kaluza and Klein.

- Newton-Cartan spacetime arises from a reduction on a null translation as shown by Duval et al. [*]

- Carrollian spacetime arises as the pull-back to a null hyperplane. Indeed given any null surface (like future null infinity $\mathcal{I}^+$) Carrollian structures come into play.

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[*] C. Duval, G. Burdet, H. P. Kunzle and M. Perrin, Bargmann Structures and Newton-cartan Theory Phys. Rev. D **31** (1985) 1841. doi:10.1103/PhysRevD.31.1841

$$
\begin{aligned}
ds^2 &= -2dudv + dx^i dx^i \qquad\qquad i = 1, \ldots, n-2 \\
p_i &= \partial_i \,, \qquad\qquad L_{ij} = x_i\partial_j - x_j\partial_i \\
U &= \partial_u \,, \qquad\qquad V = \partial_v \,, \qquad\qquad N = u\partial_i - v\partial_v \\
U_i &= u\partial_i + x_i\partial_v \,, \qquad\qquad V_i\partial_i + x_i\partial_v \\
Bargmann(n-2,1) &: \quad , \quad \mathsf{span}\{p_i, L_{ij}, U, V, U_i\} \\
Galilei(n-2,1) &: \qquad\qquad Bargmann(n,1)/V \\
Carroll(n-2,1) &: \qquad \mathsf{span}\{p_i, L_{ij}, V, U_i\}
\end{aligned}
$$

We define a Bargmann Manifold as a triple $\{B, G, \xi\}$ where B is a (d+2) manifold, G a Lorentzian metric (i.e non-degenerate and signature $(d+1, 1)$ and a null vector field ξ which is parallel w.r.t. the Levi-Civita connection of G. The standard flat Bargmann structure is given by $B =$, $\xi = \frac{\partial}{\partial s}$ with

$$ds^2 = \delta_{ij} dx^i dx^j + 2dtds$$

Note that *both* s and t are null coordinates.

The standard flat Newton-Cartan structure is obtained by pushing forward the flat Bargmann structure to the quotient or lightlike shadow or null reduction $N = B/(\mathbb{R}\xi)$ The Bargmann group consists of those isometries of B which preserve ξ. This is a central extension of the Galilei group, the centre being generated by ξ.

One may also obtain the central extension of the conformal Schrödinger group, the symmetry of the free Schrödinger equation as the those conformal transformations of $d+2$-dimensional Minkowski spacetime which commute with the action of $\mathbb{R}\xi$.

A massless scalar field in $\mathbb{E}^{d+1,1}$ is invariant under conformal transformations

$$2\frac{\partial^2\phi}{\partial t\partial s}\phi(s,t,x^i)+\nabla^2\phi=0\,.$$

set

$$\xi\phi=-im\phi\,,\qquad \phi=e^{-ims}\Psi(t,x^i)$$

then

$$i\frac{\partial\Psi}{\partial t}=-\frac{1}{2m}\nabla^2\Psi\,.$$

The standard flat Carroll structure is obtained by pulling back the flat Bargmann structure to a null hypersurface $t =$ constant. The Carroll group consists of this isometries of B which commute with the pull back.

By a Lie-algebra co-homology argument it has been shown that that the Carroll group admits no central extension.

A non-standard Carroll structure may be obtained by taking the product $C = \mathbb{R} \times \Sigma_d$ where Σ_d with Riemmannian metric $\widehat{g}$ and $g = \widehat{g} \oplus 0 \times du^2$ and $\xi = \frac{\partial}{\partial u}$, where u is a coordinate on $\mathbb{R}$. For ∇ we *could* take the Levi-civita connection of $\{\Sigma, \widehat{g}\}$.

For a general Carroll structure $\{C, g, \xi \nabla\}$ we define the Conformal Carroll group of level N as consisting of diffomeorphisms a such that

$$a^{\star}\widehat{g} = \Omega^{2}\widehat{g}\,, \qquad a_{\star} = \Omega^{-\frac{2}{N}}\xi$$

For the flat Carroll structure this has Killing vactor fields

$$X = (\omega_{ij}x_j + \gamma_i(\chi - 2\kappa_i x_i) + \kappa_i x_j x_j)\frac{\partial}{\partial x_i} + \left(\frac{2}{N}(\chi - 2\kappa_j x_j)u + T(x_k)\right)\frac{\partial}{\partial u}$$

This is *infinite dimensional* because of the super-translations $T(x_i)$ which have conformal weight $= -\frac{2}{N}$, i.e. are densities of weight $\nu = -\frac{2}{Nd}$. The quantity $z = \frac{2}{N}$ is known as a dynamical exponent.

If $N=2$, $z=1$ and we have symmetry between the scaling of space and time.

If $d=1$, using the isomorphism between the Carroll and Galilei algebras described above we obtain the Conformal Galilei algebra. CGA introduced by many people in a variety of contexts.

The isometry group of the flat Carroll structure is obtained by setting $\Omega=1$. Its Lie algebra is also infinite dimensional, because of the supertanslations. Requiring that the connection is preserved reduces the Carroll Lie algebra to the standard finite dimensonal case obtained by Levy-Lebond and Bacry.

Example

If $\{\Sigma_d, \hat{g}\} = \{S^1, d\theta^2\}$ we get Diff(S^1) semi-direct product super translations of weight $\nu = -\frac{2}{N}$ generated by the vector field

$$X = Y(\theta)\frac{\partial}{\partial\theta} + \Big(\frac{2}{N}Y'(\theta) + T(\theta)\Big)\frac{\partial}{\partial u}\,.$$

whose algebra is an extension of the Witt or Virasoro algebra.

Example

If $\{\Sigma_d, g\} = \{S^2, d\theta^2 + \sin^2\theta d\phi^2\}$ and $N = 2$ we get

$$PSL(2,\mathbb{C}) \ltimes \mathcal{T}$$

where $\mathcal{T}$ are half densities on S^2 which is the Bondi-Metzner-Sachs Group

Which was originally discovered as the asymptotic symmetry group of an asymptotically flat four-dimensional spacetime. The BMS Group has an obvious generalisation to S^d for all $d > 2$. However this generalistion does not appear to coincide with the asymptotic symmetry group of an asymptotically flat spacetime of dimension greater than four.

- Carrollian and BMS symmetries have a number of applications to various topics of current interest to string theorists and holographers which was part of the original motivation for the work reported here.

- Carrollian symmetries arise in tachyon condensates and the strong coupling limits of Born-Infeld theories on brane. * .

- Using our enhanced understanding of the Carroll group we were able construct Carrollian-invariant theories of electromagnetism which potentially have applications to slow light.

- Using a geometric quantization method of Souriau we constructed theories of Carrollian massive and massless particles. One finds the former do not move, consistent with other view points.

*G. Gibbons, K. Hashimoto and P. Yi, Tachyon condensates, Carrollian contraction of Lorentz group, and fundamental strings,'' JHEP **0209** (2002) 061

- One of the most intriguing was to Schild or Null Strings, that is strings whose two-dimensional world sheet carries a Carrollian metric, i.e is a two-dimensional null surface. It turns out that Souriau's procedure for obtaining dynamical systems invariant under a group G applied to massless "particles " leads to Schild Strings.

We turn now to Carroll Symmetry of Plane gravitational waves and to Soft Gravitons & the Memory Effect for Plane Gravitational waves. Plane gravitational are exact vacuum metrics with a covariantly constant null Killing vector admitting a five-dimensional isometry group acting on null hypersurfaces containing a three-dimensional abelian subgroup. The isometry group is a 5-dimensional subgroup of the 6 dimensional Carroll group in 2+1 dimensions with the rotations broken. As shown my Penrose they are not gobally hyperbolic but are nevertheless as useful a model palne elctromagetic waves

These solutions have many remarkable properties.

- They admit a covariantly constant spinor field. They are thus BPS.
- All invariants constructed from the Rieman tensor and its covariant derivatives
- They suffer no quantum corrections
- They are exact solutions of string theory

These metrics admit two useful coordinates systems.

Brinkmann coordinates which are global

$$ds^2 = \delta_{ij}\, dX^i dX^j + 2dU dV + K_{ij}(U) X^i X^j\, dU^2\,, \qquad \mathrm{Tr} K = 0\,.$$

Baldwin-Jeffery-Rosen Coordinates which *always* have coordinate singularities

$$ds^2 = a_{ij}(u)\, dx^i dx^j + 2du\, dv$$

$$\mathbf{X} = P(u)\mathbf{x}\,, \qquad U = u\,, \qquad V = v - \frac{1}{4}\mathbf{x}\cdot a(u)\mathbf{x}$$

$$a = P^t P\,, \qquad \ddot{P} = KP\,, \qquad P^t\dot{P} - \dot{P}^t P = 0\,.$$

$$K = \frac{1}{2} P\Big(\dot{b} + \frac{1}{2} b^2\Big) P^{-1}, \qquad b = a^{-1}\dot{a}.$$

In Brinkmann coordinates the field equations are trivially satisfied

$$K_{11} = -K_{22} = \frac{1}{2}\mathcal{A}_{+}(U)\,, \qquad K_{12} = K_{21} = \frac{1}{2}\mathcal{A}_{\times}(U)$$

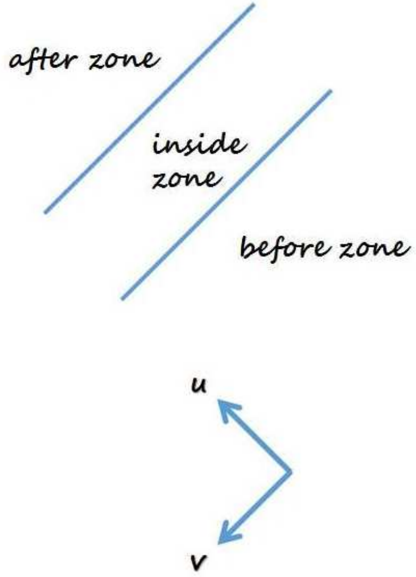
where the two polarization amplitudes may be given as arbitrary functions of U.

In Baldwin-Jeffery-Rosen coordinates the field equations are highly non-linear and even the flat solutions are non-trivial.

However the high degree of manifest symmetry in Baldwin-Jeffery-Rosen coordinates allows exact solution of all geodesics

$$\frac{dx^i}{du} = a^{ij} p_j$$

Consider a sandwich wave[*]



passing over a cloud of particles all at relative rest before the wave arrives.

[*] for which $\mathcal{A}_+(U)$ and $\mathcal{A}_\times(U)$ vanish outside a finite interval $U_i \leq U \leq U_f$.

By Noether's theorem the Baldwin-Jeffery-Rosen coordinates $\mathbf{x}$ are constant both before and after the wave has passed

But even if $a_{ij} = \delta_{ij}$ before the wave has arrived a_{ij} will, in general have non-trivial time dependence after the wave has passed. Thus the separations of the particles will in general have non-trivial time dependence after the wave has passed. This may in principle be measured and information about $\mathcal{A}_+(U)$ and $\mathcal{A}_\times(U)$ deduced.

THIS IS THE GRAVITATIONAL MEMORY EFFECT

This is sometimes crassly referred to as “a permanent change of spacetime after the wave has passed.”. It it is no such thing.

In a metric of the form

$$ds^2 = -dt^2 + g(x,t)_{ab} dx^a dx^b$$

solutions of Maxwell's equations behave as if in an impedance matched medium in flat spacetime with

$$\epsilon_{ab} = \mu_{ab} = \sqrt{\det g}(g^{-1})_{ab}$$

In BJR coordinates

$$\epsilon_{ij} = \sqrt{\det a}(a^{-1})_{ij}\,, \qquad \epsilon_{33} = \sqrt{\det a}$$

After the wave has passed although $a_{ij} \neq \delta_{ij}$ there is a coordinate transformation which we calculate explicitly, which brings the metric after the wave has passed to canonical flat form. This coordinate transformation does not tend to the identity at spatial infinity.

THUS FLAT PLANE WAVE VACCUUM METRICS IN BALDWIN-JEFFERY-ROSEN COORDINATES CAN BE THOUGHT OF AS SOFT GRAVITONS LEFT AFTER THE PASSING OF A WAVE PULSE WITH NON-VANISHING CURVATURE

One may give many reasons why Stephen and I did not discover the memory effect

- We were too stupid and but this is obviously wrong
- We were considering bar detectors and Zeldovich and Polanarev's argument does not apply
- No one had thought or built interferomometer detectors let alone had dreamt of satellite detectors
- Even if we had thought about it we were only using linear theory and so any argument would have been unconvincing
- There is no such effect

IN FACT THE LAST TWO BULLET POINTS ARE TRUE!

The vanishing of the Ricci tensor leads to the *exact* equation

$$\mathrm{Tr}\big(P(\dot{b}+\frac{1}{2}b^2)P^{-1}\big)=0, \qquad b=a^{-1}\dot{a}$$

This is a form of the Raychaudhuri equation Standard methods show that there are no solutions which start at rest and end at rest. A result known to Bondi and Pirani.

Thus, *sensu stricto* there is no memory effect in the sense originally stated by Zeldovich and Polanarev and numerical calculations bear this out.

However no gravitational wave is *exactly plane* and even if focussing does lead to a caustic, this is likely to take a very long time in a solar system size detector such as LISA or even terrestial detector such as LIGO.

This talk was based on the following papers

P. M. Zhang, C. Duval, G. W. Gibbons and P. A. Horvathy, Soft Gravitons and the Memory Effect for Plane Gravitational Waves `To appear`

P. M. Zhang, C. Duval, G. W. Gibbons and P. A. Horvathy, The Memory Effect for Plane Gravitational Waves, `arXiv:1704.05997 [gr-qc].`

C. Duval, G. W. Gibbons, P. A. Horvathy and P.-M. Zhang, Carroll symmetry of plane gravitational waves," `arXiv:1702.08284 [gr-qc].`

C. Duval, G. W. Gibbons and P. A. Horvathy Conformal Carroll groups, J. Phys. A **47** (2014) 335204 `[arXiv:1403.4213 [hep-th]]`

C. Duval, G. W. Gibbons and P. A. Horvathy, 'Conformal Carroll groups and BMS symmetry,' Class. Quant. Grav. **31** (2014) 092001 `[arXiv:1402.5894 [gr-qc]]`

C. Duval, G. W. Gibbons, P. A. Horvathy and P. M. Zhang, 'Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time, Class. Quant. Grav. **31** (2014) 085016 `[arXiv:1402.0657 [gr-qc]]`

C. Duval, G. W. Gibbons and P. Horvathy, Celestial mechanics, conformal structures and gravitational waves, Phys. Rev. D **43** (1991) 3907 `[hep-th/0512188].`