Quasinormal modes of AdS black holes

Claude Warnick

University of Warwick

Cambridge, March 2014

Based on 1306.5760

Goals

- Want to define quasinormal modes for general stationary AdS black hole spacetimes
  - Avoid symmetry assumptions (in particular no separability)
  - Avoid analyticity assumptions
- Want to understand the completeness (or otherwise) of the quasinormal mode spectrum
  - To what extent is a perturbation captured by its QNM spectrum
- I will restrict attention to the well understood case of a scalar field on Schwarzschild-AdS, but full theorem is much more general.

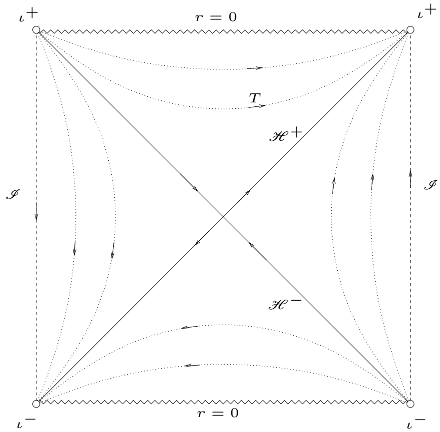
What are quasinormal modes?

- Quasinormal modes are *characteristic oscillations* of linear fields on black hole backgrounds.
- They are time harmonic solutions of the field equations

$$\psi \propto e^{st},$$

- They both oscillate and decay: the *quasinormal frequencies*, s, are complex.
- Play a role in late time behaviour of the scalar field.

The Schwarzschild-AdS Spacetime



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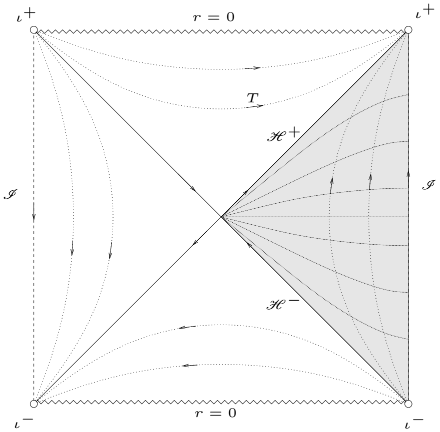
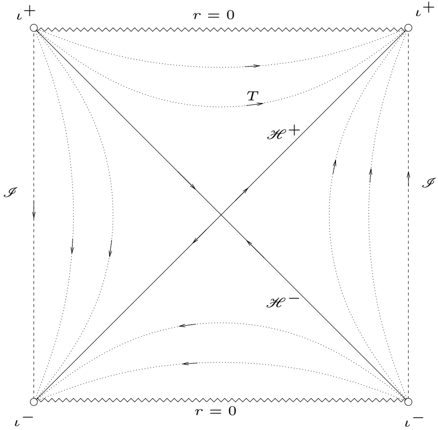


$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{r^2}{l^2}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{r^2}{l^2}} + r^2 d\Omega^2$$

The regular slicing



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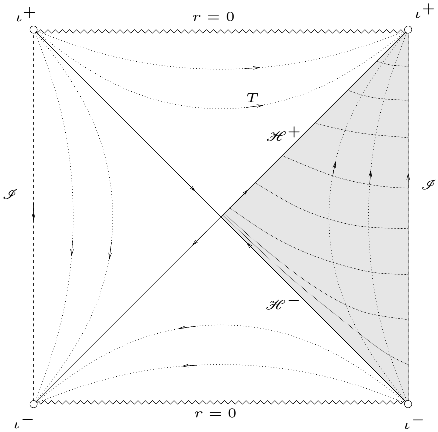


$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{r^2}{l^2}\right) dt^2 + \frac{4M}{r\left(1 + \frac{r^2}{l^2}\right)} dt dr + \frac{1 + \frac{2M}{r} + \frac{r^2}{l^2}}{\left(1 + \frac{r^2}{l^2}\right)^2} dr^2 + r^2 d\Omega^2$$

Quasinormal modes as eigenvalues

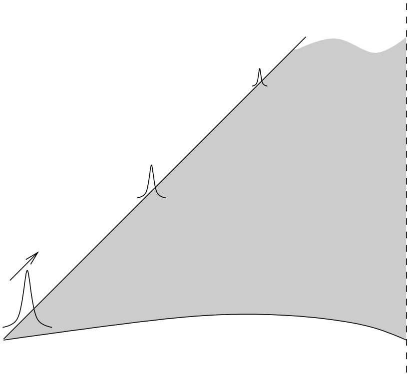
- Consider the conformally coupled Klein-Gordon equation:

$$\Box_g \psi - \frac{2}{l^2}\psi = 0$$
$$\psi|_{t=0} = \uppsi, \quad \partial_t \psi|_{t=0} = \uppsi',$$
$$r\psi \to 0, \text{ as } r \to \infty.$$

- Solution exists for all $t \geq 0$.
- Want to understand late time behaviour of solutions to this equation
  - In particular, we are interested in features characteristic of the spacetime (not of particular choices of initial data)
- Regularity at horizon is an important factor

[Horowitz–Hubeny; Bizon et al.]

Wavepackets at the horizon



Wavepackets at the horizon

- Rate of decay determined by how sharply localised the wave packet is
- Measure localisation using Sobolev norms
- For a function $u(x)$ defined on $\mathbb{R}^n$, with Fourier transform $\tilde{u}(\xi)$, define

$$||u(x)||^2_{H^k} = \int d^n\xi (1 + |\xi|^2)^k |\tilde{u}(\xi)|^2$$

- Can extend definition to curved manifolds
- The larger k is, the smoother a function with $||u(x)||_{H^k} < \infty$ is.
- Crudely, an outgoing wavepacket localised at the horizon, with $||\psi(x,t)||_{H^k} < \infty$ will decay like

$$|\psi(x,t)| \sim e^{-\varkappa(k-\frac{1}{2})}$$

where $\varkappa$ is the surface gravity.

What are quasinormal modes? Version 2

- Solutions of $\Box_g \psi - \frac{2}{l^2}\psi = 0$ satisfy

$$||\psi||^2_{H^k(\Sigma_t)} + ||T\psi||^2_{H^{k-1}(\Sigma_t)} \leq C \left(||\psi||^2_{H^k(\Sigma_0)} + ||T\psi||^2_{H^{k-1}(\Sigma_0)} \right)$$

[Holzegel; Holzegel–CMW; Dafermos–Rodnianski]

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- The map

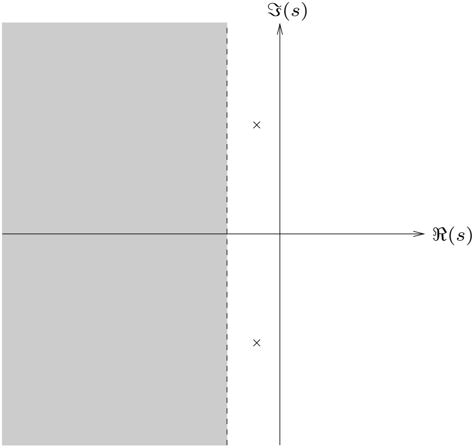
$$\begin{array}{rccc} \mathcal{S}(t) & : & \boldsymbol{H}^k(\Sigma) & \to & \boldsymbol{H}^k(\Sigma) \\ & & (\psi, T\psi)|_{\Sigma_0} & \mapsto & (\psi, T\psi)|_{\Sigma_t} \end{array}$$

is a C^0-semigroup, so can write:

$$\mathcal{S}(t) = e^{t\mathcal{A}}$$

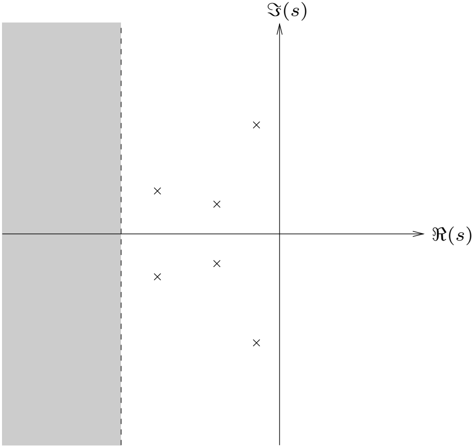
Here $\mathcal{A}$ is a (degenerate) elliptic operator on Σ.
[c.f. $U(t) = e^{it\Delta}$ for Scrödinger equation on $\mathbb{R}^n$].

The spectrum of $(D^k(\mathcal{A}), \mathcal{A})$



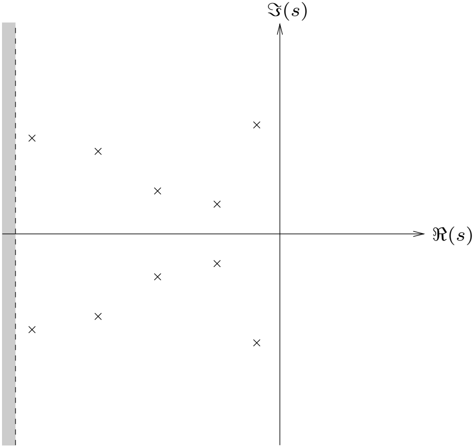
$k = 1$

THE SPECTRUM OF $(D^k(\mathcal{A}), \mathcal{A})$



$$k = 2$$

The spectrum of $(D^k(\mathcal{A}), \mathcal{A})$



$k = 3$

The main theorem

Theorem (Discreteness of QNF [CMW, 2013])

The spectrum of $\mathcal{A}$ in the region $\Re(s) > \left(\frac{1}{2} - k\right)\varkappa$ consists solely of isolated eigenvalues of finite multiplicity. The eigenfunctions u are smooth at the horizon and if $\psi = e^{st}u$, we have

$$\Box_g \psi - \frac{2}{l^2}\psi = 0.$$

Related work: [Horowitz–Hubeny; Vasy; Bachelot; Gannot; Melrose–Sá Baretto–Vasy; Dyatlov; Sá Baretto–Zworski; Bony–Häfner; ...]

Corollary

Let $\psi(x,t)$ be a smooth solution of the Klein-Gordon equation on an asymptotically AdS black hole. Then the Laplace transform

$$\hat{\psi}(x,s) = \int_0^\infty e^{-st}\psi(x,t)dt$$

extends meromorphically to $\mathbb{C}$, and the location of its poles belong to a countable set Λ_{QNF} which is independent of ψ.

Corollary

Suppose there exist QNM with $|\Im(s_n)| \to \infty$ *as* $n \to \infty$ *and such that for some* C

$$-C\left(\Im(s_n)\right)^{-\frac{1}{\alpha}} < \Re(s_n) \leq 0.$$

Then for any $\epsilon > 0$*, there exists a solution* ψ *with initial data in* $D^1(\mathcal{A})$ *such that*

$$||\psi||_{\underline{H}^1(\Sigma_t)} + ||T\psi||_{\underline{L}^2(\Sigma_t)} \geq \frac{1}{t^{\alpha+\epsilon}}, \qquad \text{as } t \to \infty.$$

The main theorem

- No separability of the equations is assumed
- Regularity as a boundary condition is very natural
- Can extend to any other of the usual linear fields (Dirac, Maxwell, etc.)
- Can extend to arbitrary locally stationary black holes
- Unlike the usual definition using 'ingoing' boundary conditions, QNM are honest eigenfunctions of an operator on a Hilbert space
- Do not need to restrict to perturbations supported away from the horizon
- Can show that 'ingoing' QNF are a subset of these QNF, and they typically agree

Completeness of the spectrum

- Since QNF spectrum is countable, is it true by analogy with Fourier series that if ψ is a solution of KGE, then

$$\psi(x,t) = \sum_{i=0}^{\infty} a_i e^{s_i t} u_i(x)?$$

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- In fact, can arrange that $\sum_{i=0}^{\infty} e^{s_i t} u_i(x)$ converges,

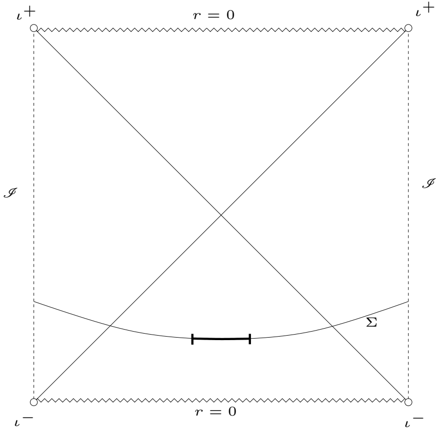
$$\psi(x,t) \sim \sum_{i=0}^{\infty} a_i e^{s_i t} u_i(x) \qquad \text{as } t \to \infty$$

  but nevertheless

$$\psi(x,t) \neq \sum_{i=0}^{\infty} e^{s_i t} u_i(x)$$

  for *any* finite t.

Incompleteness for AdS Schwarzschild



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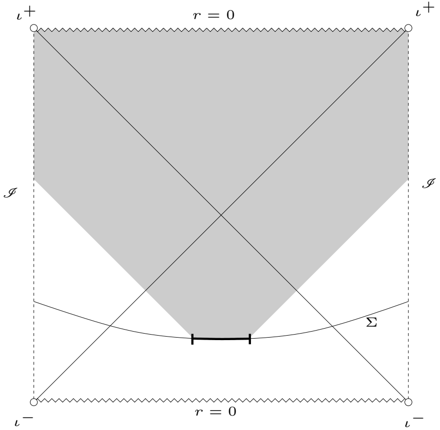

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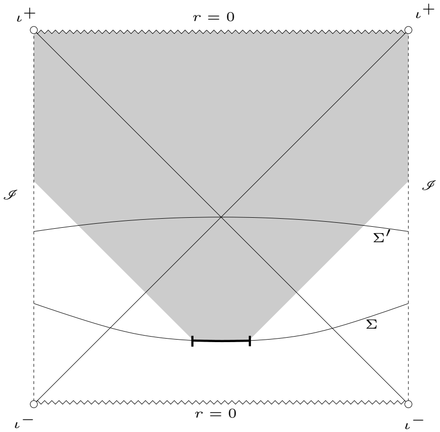

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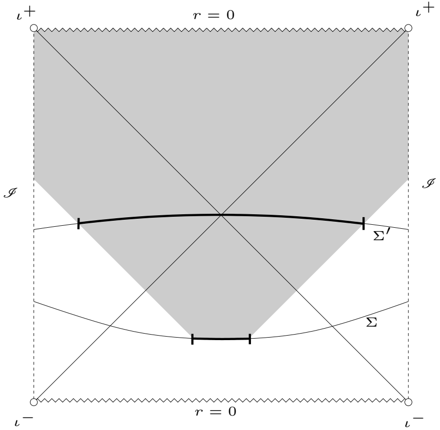

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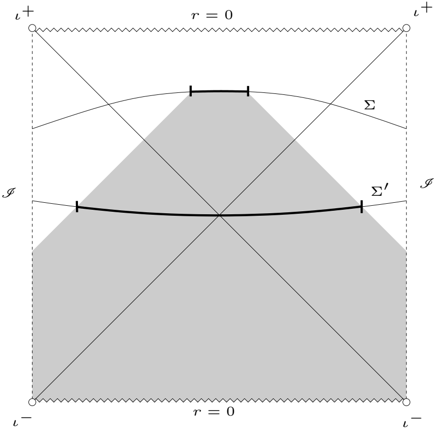

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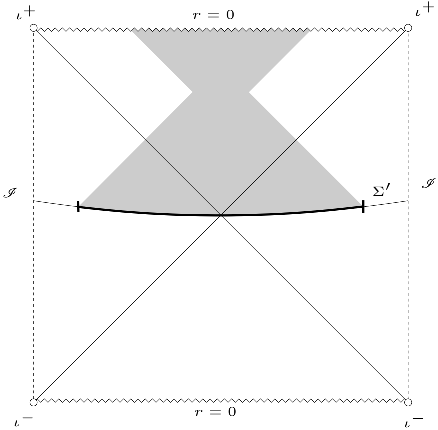

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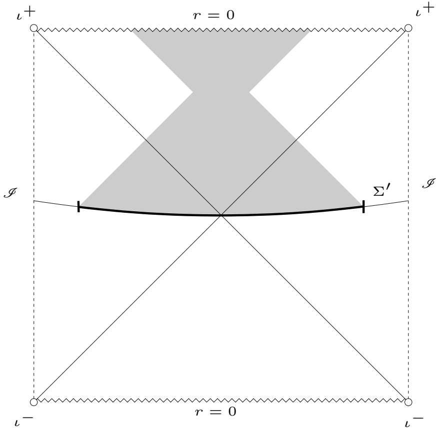

$$\psi \sim 0 \quad \text{as } t \to \infty, \qquad \text{but } \psi \not\equiv 0.$$

Conclusions

- QNM should be thought of as eigenvalues of the infinitesimal generator of the solution operator on $H^k \times H^{k-1}$ for a regular slicing
- The QNF are a discrete, countable set of points in the complex plane
- The QNM do not form a complete basis for $H^k \times H^{k-1}$.