Non-equilibrium dynamics and the Robinson-Trautman solution

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New Frontiers in Dynamical Gravity
Cambridge, UK, 28 March 2014

Introduction

- Gauge/gravity duality offers a new tool to study non-equilibrium dynamics at strong coupling.
- AdS black holes correspond to thermal states of the CFT.
- Black hole formation corresponds to thermalization.

Introduction

- Hydrodynamics capture the dynamics the long wave-length, late time behavior of QFTs close to thermal equilibrium.
- On the gravitational side, one can construct bulk solutions in a gradient expansion that describe the hydrodynamic regime.
- Global solutions corresponding to non-equilibrium configurations should be well-approximated by the solutions describing the hydrodynamic regime at sufficiently long distances and late times.

Introduction

- Almost all work on global solutions is numerical.
- In this work we aim at obtaining analytic solutions describing out-of-equilibrium dynamics.
- We will discuss this in the context AdS_4/CFT_3.

References

- This talk is based on work done with I. Bakas, *to appear*.
- Related work appeared very recently in [G. de Freitas, H. Reall, 1403.3537]

Equilibrium configuration

- The thermal state corresponds to the AdS Schwarzschild black hole

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

  with $f(r) = 1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2$.
- Linear perturbations around the Schwarzschild solution describe holographically thermal 2-point functions in the dual QFT.
- From those, using linear response theory, one can obtain the transport coefficients entering the hydrodynamic description close to thermal equilibrium.
- To describe out-of-equilibrium dynamics we need to go beyond linear perturbations.

Strategy

To describe analytically non-equilibrium phenomena and their approach to equilibrium we need

- Exact time-dependent solutions of Einstein equations.
- These solutions should limit at late times to the Schwarzschild solution.

- Can we find analytically exact solutions corresponding to linear perturbations of the Schwarzschild solution?

Linear perturbations of AdS Schwarzschild

Parity even metric perturbations of Schwarzschild solution are parametrized by

$$\begin{pmatrix} f(r)H_0(r) & H_1(r) & 0 & 0 \\ H_1(r) & H_0(r)/f(r) & 0 & 0 \\ 0 & 0 & r^2K(r) & 0 \\ 0 & 0 & 0 & r^2K(r)\sin^2\theta \end{pmatrix} e^{-i\omega t}P_l(\cos\theta)\,,$$

where $P_l(\cos\theta)$ are Legendre polynomials. (For simplicity we only display axially symmetric perturbations.)

- There are also parity odd perturbations. We will not need their explicit form here.

Effective Schödinger problem

- The study of these perturbations can be reduced to an effective Schrödinger problem [Regge, Wheeler] [Zerilli] ...

$$\left(-\frac{d^2}{dr_\star^2} + W^2 \pm \frac{dW}{dr_\star}\right)\Psi(r_\star) = E\,\Psi(r_\star)\ .$$

- The two signs correspond to the parity even and odd cases.
- $E = \omega^2 - \omega_{\rm s}^2, \qquad \omega_{\rm s} = -\frac{i}{12m}(l-1)l(l+1)(l+2)\ .$
- $\Psi_{even}(r) = \frac{r^2}{(l-1)(l+2)r+6m}\left(K(r) - i\frac{f(r)}{\omega r}H_1(r)\right)$ and there is a similar formula for the odd case.
- $r_\star$ is the tortoise radial coordinate, $dr_\star = dr/f(r)$.
- $W(r) = \frac{6mf(r)}{r[(l-1)(l+2)r+6m]} + i\omega_{\rm s}$

Supersymmetric Quantum mechanics

- There is an underlying supersymmetric structure with W being the superpotential,

$$H_{even} = Q^\dagger Q + \omega_s^2, \qquad H_{odd} = QQ^\dagger + \omega_s^2$$

with

$$Q = \left(-\frac{d}{dr_\star} + W(r_\star)\right), \qquad Q^\dagger = \left(\frac{d}{dr_\star} + W(r_\star)\right)$$

- Forming

$$H = \begin{pmatrix} H_{even} & 0 \\ 0 & H_{odd} \end{pmatrix} \quad \mathbf{Q} = \begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix}$$

one finds that they form a SUSY algebra, $\{\mathbf{Q}, \mathbf{Q}^\dagger\} = H$ etc.

Remarks

- The Hamiltonian is only formally hermitian.
- Boundary condition break supersymmetry.
- E is not bounded from below, it is not even real.

Zero energy solutions

- A special class of solutions are those with zero energy,

$$E = 0 \qquad \Leftrightarrow \qquad \omega = \omega_s$$

- These modes satisfy a first order equation

$$Q\Psi_0 = \left(-\frac{d}{dr_\star} + W(r_\star)\right)\Psi_0 = 0$$

  They are the supersymmetric ground states of supersymmetric quantum mechanics.
- These are the so-called algebraically special modes [Chandrasekhar].
- It is these modes that we would like to study at the non-linear level.

Boundary conditions

- Ψ_0 vanishes at the horizon.
- It is finite and satisfies mixed boundary conditions at the conformal boundary,

$$\frac{d}{dr_\star}\Psi_0(r_\star)\mid_{r_\star=0} = \left(i\omega_{\rm s} - \frac{2m\Lambda}{(l-1)(l+2)}\right)\Psi_0(r_\star=0)\ .$$

- It is normalizable,

$$\int_{-\infty}^{0} dr_\star \mid \Psi_0(r_\star)\mid^2 < \infty\ .$$

Robinson-Trautman spacetimes

- The metric is given by

$$ds^2 = 2r^2 e^{\Phi(z,\bar{z};u)} dz d\bar{z} - 2du dr - F(r,u,z,\bar{z}) du^2$$

- The function F is uniquely determined in terms of Φ,

$$F = r\partial_u \Phi - \Delta\Phi - \frac{2m}{r} - \frac{\Lambda}{3} r^2$$

  where Λ is related to the cosmological constant and $\Delta = e^{\Phi}\partial_z \partial_{\bar{z}}$.
- The function $\Phi(z,\bar{z};u)$ should solve the following *Robinson-Trautman equation*,

$$3m\partial_u \Phi + \Delta\Delta\Phi = 0.$$

Robinson-Trautman equation and the Calabi flow

- The Robinson-Trautman equation coincides with the Calabi flow on S^2 that describes a class of deformations of the metric

$$ds_2^2 = 2e^{\Phi(z,\bar{z};u)} dz d\bar{z} \ .$$

- The Calabi flow is defined more generally for a metric $g_{a\bar{b}}$ on a Kähler manifold M by the Calabi equation

$$\partial_u g_{a\bar{b}} = \frac{\partial^2 R}{\partial z^a \partial z^{\bar{b}}}$$

where R is the curvature scalar of g.

- It provides volume preserving deformations within a given Kähler class of the metric.

Calabi flow on S^2

- The Calabi flow can be regarded as a non-linear diffusion process on S^2.
- Starting from a general initial metric $g_{a\bar{b}}(z, \bar{z}; 0)$, the flow monotonically deforms the metric to the constant curvature metric on S^2, described by

$$e^{\Phi_0} = \frac{1}{(1 + z\bar{z}/2)^2} \ .$$

AdS Schwarzschild as Robinson-Trautman

- Using the fixed point solution of the Robinson-Trautman equation

$$e^{\Phi_0} = \frac{1}{(1 + z\bar{z}/2)^2} .$$

the metric becomes

$$ds^2 = \frac{2r^2}{(1 + z\bar{z}/2)^2} dz d\bar{z} - 2dudr - \left(1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2\right) du^2$$

which is the Schwarzschild metric in the Eddington - Filkenstein coordinates.

Zero energy solutions as Robinson-Trautman

- Perturbatively solving the Robinson-Trautman equation around the round sphere

$$ds_2^2 = [1 + \epsilon_l(u) P_l(\cos\theta)] \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

  one finds

$$\epsilon_l(u) = \epsilon_l(0) e^{-i\omega_s u}$$

  with

$$\omega_s = -i\frac{(l-1)l(l+1)(l+2)}{12m}$$

- This is exactly the frequency of the zero energy solutions we found earlier!
- Inserting in the Robinson-Trautman metric we find the zero energy perturbations of AdS Schwarzschild we discussed earlier.

Summary

The Robinson-Trautman solution is a non-linear version of the algebraically special perturbations of Schwarzschild.

Late-time behavior of solutions [Chruściel, Singleton]

- We parametrize the conformal factor of the S^2 line element as

$$e^{\Phi(z,\bar{z};u)} = \frac{1}{\sigma^2(z,\bar{z};u)\,(1+z\bar{z}/2)^2}\,.$$

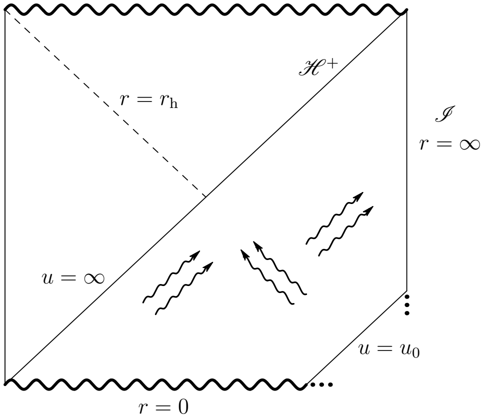
- $\sigma(z,\bar{z};u)$ has the following asymptotic expansion

$$1+\sigma_{1,0}(z,\bar{z})e^{-2u/m}+\sigma_{2,0}(z,\bar{z})e^{-4u/m}+\cdots+\sigma_{14,0}(z,\bar{z})e^{-28u/m}$$
$$+[\sigma_{15,0}(z,\bar{z})+\sigma_{15,1}(z,\bar{z})u]e^{-30u/m}+\mathcal{O}\left(e^{-32u/m}\right).$$

- The terms with $\sigma_{1,0},\sigma_{5,0},\sigma_{15,0},\ldots$ are due to the linear algebraically special modes with $l=2,3,4,\ldots$.
- The other terms are due to non-linear effects.

Global aspects

For large black holes, the solution does not appear to have a smooth extension beyond $u \to \infty$ [Bicak, Podolsky].



Other properties

- There is a past apparent horizon Σ, whose position $r = U(z, \bar{z})$ and area $\mathrm{Area}(\Sigma)$ we determined.
- At late times, $\mathrm{Area}(\Sigma)$ decreases and becomes equal to area of the Schwarzschild horizon as $u \to \infty$.
- One can define a Bondi mass $\mathcal{M}_{\mathrm{Bondi}}$ with the properties

$$\mathcal{M}_{\mathrm{Bondi}} \geq m, \quad \frac{d}{du}\mathcal{M}_{\mathrm{Bondi}} \leq 0,$$

that satisfies a Penrose inequality

$$16\pi\mathcal{M}^2_{\mathrm{Bondi}} \geq \mathrm{Area}(\Sigma)\left(1 - \frac{\Lambda}{3}\frac{\mathrm{Area}(\Sigma)}{4\pi}\right)^2 .$$

Holography

- The boundary metric is time-dependent and it is not conformally flat

$$ds^2_{\mathscr{I}} = -dt^2 - \frac{6}{\Lambda} e^{\hat{\Phi}(z,\bar{z};t)} dz d\bar{z} \, .$$

  where $\hat{\Phi}(z,\bar{z};t) = \Phi(z,\bar{z};u = t - r^\star)|_{r^\star=0}$.
- The holographic energy momentum tensor is

$$\kappa^2 T^{\text{ren}}_{tt} = -\frac{2m\Lambda}{3} \, , \qquad \kappa^2 T^{\text{ren}}_{tz} = -\frac{1}{2}\partial_z(\hat{\Delta}\hat{\Phi})$$

$$\kappa^2 T^{\text{ren}}_{z\bar{z}} = m e^{\hat{\Phi}} \, , \qquad \kappa^2 T^{\text{ren}}_{zz} = -\frac{3}{4\Lambda}\partial_t \left((\partial_z \hat{\Phi})^2 - 2\partial_z^2 \hat{\Phi} \right) ,$$

Algebraically special modes

- The holographically energy momentum tensor for the linear algebraically special modes can be rewritten in a fluid form

$$T^{ab} = \rho u^a u^b + p\Delta^{ab} - \eta\sigma^{ab}$$

- 3-velocity

$$u_t = -1, \quad u_\phi = 0, \quad u_\theta = \frac{1}{4m\Lambda}(l-1)(l+2)e^{-i\omega_s t}\partial_\theta P_l(\cos\theta)$$

- viscosity

$$\kappa^2\eta = \frac{1}{4}l(l+1)$$

Violation of KSS bound

$$\frac{\eta}{s} = \frac{l(l+1)}{8\pi} \frac{r_{\rm h}}{2m - r_{\rm h}}$$

- The bound $\eta/s \geq 1/4\pi$ is violated for large black holes and small enough l.
- These modes however do not satisfy Dirichlet boundary conditions.
- All modes that violate the bound do not extend smoothly beyond $u = \infty$ (however there are modes that do not have smooth extension but nevertheless satisfy the bound).

Conclusions/Outlook

- The Robinson-Trautman solution is a non-linear version of the algebraically special perturbation of Schwarzschild.
- One can study quantitatively and analytically the approach to equilibrium and the effects of non-linear terms.
- It would be interesting to understand better holography for these solutions, in particular the implications of the unusual boundary conditions, the holographic meaning of the Bondi mass, the Penrose inequality, etc. ...