Turbulence in black holes and back again

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Motivation…

Holography provides a remarkable framework to connect gravitational phenomena in d+1 dimensions with field theories in d dimensions.

- Most robustly established between AdS ← → N=4SYM

…. the use [or 'abuse'?] of AdS in AdS/CFT… (~ 2011)

Stability of AdS?

Stability of BHs in asympt AdS?

Do we know all QNMs for stationary BHs in AdS?

- Are these a basis?
- *Linearization stability?*

Motivation…

…. the use [or ‘abuse’?] of AdS in AdS/CFT…

Stability of AdS? [No, but with islands of stability or the other way around? (see Bizon,Liebling,Maliborski)]

Stability of BHs in asympt AdS? [don’t know… but arguments against (see Holzegel)]

QNMs for stationary BHs in AdS
[(Dias,Santos,Hartnett,Cardoso,LL)]

Are these a basis? [No (see Warnick)]

Linearization stability? [No …]

Turbulence (in hydrodynamics)

or "that phenomena you know is there when you see it"

For Navier-Stokes (incompressible case):

- Breaks symmetry (back in a 'statistical sense')
- Exponential growth of (some) modes [not linearly-stable]
- Global norm (non-driven system): Exponential decay possibly followed by power law, then exponential
- Energy cascade (direct d>3, inverse/direct d=2)
- Occurring if Reynolds number is sufficiently high
- $E(k) \sim k^{-p}$ (5/3 and 3 for 2+1)
- Correlations: $< v(r)^3 > \sim r$

‘Turbulence’ in gravity?

- Does it exist? (arguments against it, mainly in 4d)
  - Perturbation theory (e.g. QNMs, no tail followed by QNM)
  - Numerical simulations (e.g. ‘scale’ bounded)
  - (hydro has shocks/turbulence, GR no shocks)

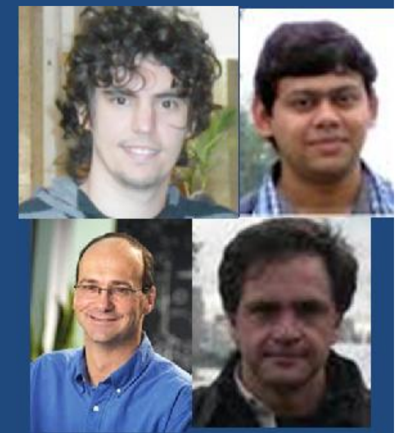
* AdS/CFT <-> AdS/Hydro (→ turbulence?! [Van Raamsdonk 08])

  - Applicable if LT >> 1 → L (ρ /ν>)> 1 → L (ρ /ν v)= Re >> 1
  - (also cascade in ‘pure’ AdS)

- List of questions?
  - Does it happen? (tension in the correspondence or gravity?)
  - Reconcile with QNMs expectation (and perturb theory?)
  - Does it have similar properties?
  - What’s the analogue `gravitational’ Reynolds number?

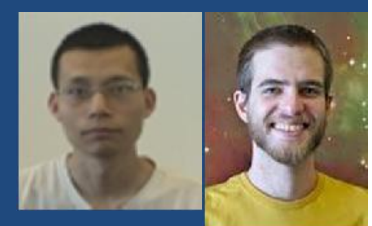
Tale of 3 1/2 projects

- Does turbulence occur in relativistic, conformal fluids ($p=\rho/d$) ? Does it have inverse cascade in 2+1? (PRD V86,2012)

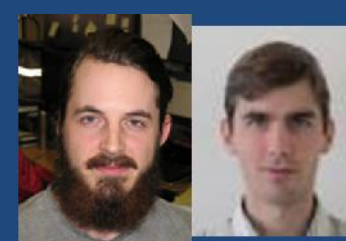


- Can we reconcile with QNM? What's key to analyze it? Intuition for gravitational analysis? (PRX, V4, 2014)

- What about in AF?, can we define it intrinsically in GR? Observables? arXiv:1402:4859



- Relativistic scaling and correlations? (ongoing)



[subliminal reminder: risks of perturbation theory]

- AdS/CFT → gravity/fluid correspondence [Bhattacharya,Hubeny,Minwalla,Rangamani; VanRaamsdonk; Baier,Romatschke,Son,Starinets,Stephanov]

$$ds^2_{[0]} = -2u_\mu dx^\mu dr + r^2 \left(\eta_{\mu\nu} + \frac{1}{(br)^d} u_\mu u_\nu \right) dx^\mu dx^\nu.$$

- $T_{ab} = T_{ab} = \frac{\rho}{d-1}(du_a u_b + \eta_{ab}) + \Pi_{ab}$
- Subject to :
  - $u_a u^a = -1$; $T^a{}_a = 0$; $\Pi_{ab} = -2\eta\sigma_{ab} + \cdots$
  - $\nabla_a T^{ab} = 0$.
- Do these eqns/eos give rise to turbulence?
  - Non-relativistic limit → Navier-Stokes eqn. why wouldn't they?
  - If so, NS eqns have indirect cascade for 2+1 dimensions. Why? There exists a conserved quantity: *enstrophy. Does it exist for these eqns/eos?*

Enstrophy? Assume no viscosity

- In Navier-Stokes:
  - vorticity: $w_{ab} = \partial_{[a}\, v_{b]}$.
  - Enstrophy $\int w^2\, ds$. Conserved if viscocity = 0.
- For relativistic hydro:
- $\Omega_{\mu\nu} = \nabla_{[\mu}\rho^{1/d}u_{\nu]}\,,$ $\Omega_{\mu\nu}u^{\nu} = 0\,,$ <-> EOM
- Then. $\pounds_{\lambda \mathrm{u}}\Omega = \lambda \mathrm{u}\cdot \mathrm{d}\Omega + \mathrm{d}(\lambda \mathrm{u}\cdot\Omega) = 0\,.$ → conserv. law.?
- In d=2+1 one can show: $\Omega^{\mu\alpha}\Omega_{\mu\beta} = \frac{1}{2}\Omega^2 P^{\alpha}_{\beta}\,,$
- And so: $J^{\mu} \equiv \rho^{-2/3}(\Omega^{\alpha\beta}\Omega_{\alpha\beta})u^{\mu}$, is divergenless, implying

$$Z \equiv \int_{S^2} \rho^{-2/3}\Omega^2 u^0 d\Sigma = \int_{S^2} (\omega_{\mu\nu}\omega^{\mu\nu} + \frac{1}{2}a_{\mu}a^{\mu})\gamma d\Sigma$$

[Carrasco,LL,Myers,Reula,Singh 2012]

And in the bulk?

- Duality provides the metric all through the bulk (so no need to solve EEs). Further, the boundary info has a 'trivial' extension into the radial direction. [Battacharya,etal]
- Further, hydro eqns are also obtained at timelike hypersurfaces. Take those defined by the unperturbed soln. Fluid EOS is not conformal: $p = (3\, f(r,\rho) - 1)\rho/4$
- A conserved enstrophy exists as well and $Z(r) \rightarrow Z_b$. So if turbulent, inverse cascade expected as well

Let's examine what happens for both Poincare patch & global AdS

- numerical simulations 2+1 on flat (T^2) or S^2

What's the 'practical' problem?

$$T^{[0+1+2]}_{\mu\nu} = \frac{\rho}{d-1}\left(du_\mu u_\nu + \eta_{\mu\nu}\right) + \Pi_{\mu\nu},$$

$$\sigma_{\mu\nu} \equiv \langle \partial_\mu u_\nu \rangle,$$
$$\omega_{\mu\nu} \equiv P_\mu{}^\alpha P_\nu{}^\beta \partial_{[\alpha} u_{\beta]}.$$

$$\begin{aligned}\Pi_{\mu\nu} = &-2\eta\sigma_{\mu\nu} \\ &+ 2\eta\tau_\Pi\left(\langle u^\alpha \partial_\alpha \sigma_{\mu\nu}\rangle + \frac{1}{d-1}\sigma_{\mu\nu}\partial_\alpha u^\alpha\right) + \langle \lambda_1 \sigma_{\mu\alpha}\sigma_\nu{}^\alpha + \lambda_2 \sigma_{\mu\alpha}\omega_\nu{}^\alpha + \lambda_3 \omega_{\mu\alpha}\omega_\nu{}^\alpha \rangle.\end{aligned}$$

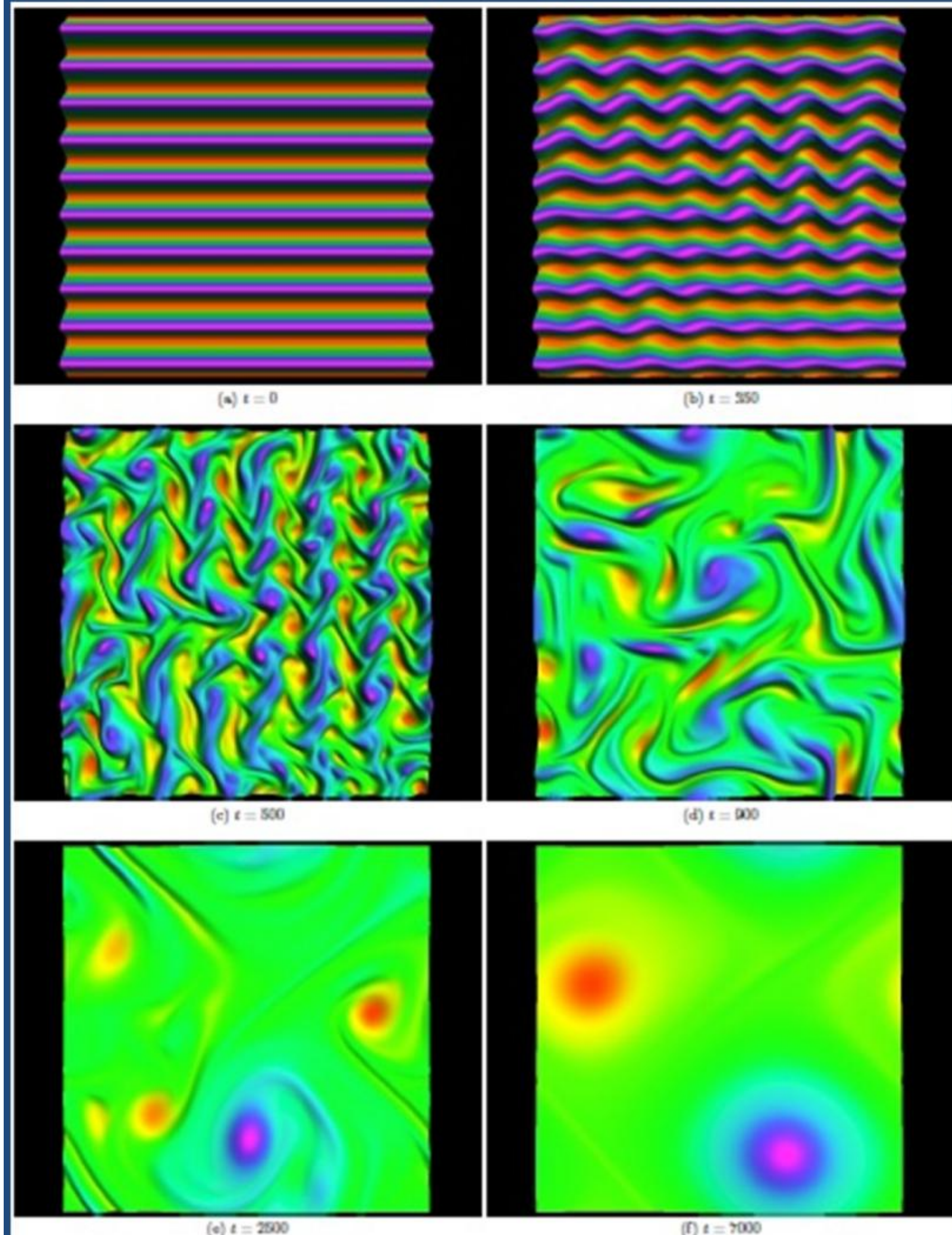
- Equations of motion

$$0 = u^\mu \partial_\mu \rho + \frac{d}{d-1}\rho \partial_\mu u^\mu - u^\mu \partial^\nu \Pi_{\mu\nu},$$

$$0 = \frac{d}{d-1}\rho u^\mu \partial_\mu u^\alpha + \frac{1}{d-1}\partial^\alpha \rho - \frac{d}{(d-1)^2} u^\alpha \rho \partial_\mu u^\mu + \frac{1}{d-1} u^\alpha u^\mu \partial^\nu \Pi_{\mu\nu} + P^{\alpha\mu}\partial^\nu \Pi_{\mu\nu}.$$

- Enforce $\Pi_{ab} \sim \sigma_{ab}$ (a la Israel-Stewart, also Geroch)

$$\begin{aligned}\Pi_{\mu\nu} = &-2\eta\sigma_{\mu\nu} \\ &- \tau_\Pi\left(\langle u^\alpha \partial_\alpha \Pi_{\mu\nu}\rangle + \frac{d}{d-1}\Pi_{\mu\nu}\partial_\alpha u^\alpha\right) + \langle \frac{\lambda_1}{\eta^2}\Pi_{\mu\alpha}\Pi_\nu{}^\alpha - \frac{\lambda_2}{\eta}\Pi_{\mu\alpha}\omega_\nu{}^\alpha + \lambda_3 \omega_{\mu\alpha}\omega_\nu{}^\alpha \rangle.\end{aligned}$$



(a) $t = 0$ (b) $t = 350$

(c) $t = 500$ (d) $t = 900$

(e) $t = 2500$ (f) $t = 7000$

Late stage → Oseen's vortex : [Gallay-Wayne: stable, attractor for NS solns in 2+1 dims]

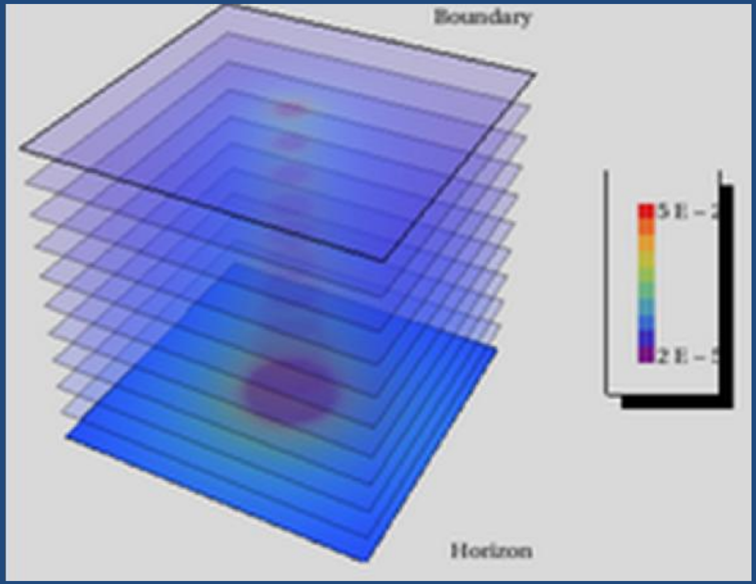
$$u_\theta = \frac{A}{r}\left(1 - e^{-r^2/4\eta t}\right)$$

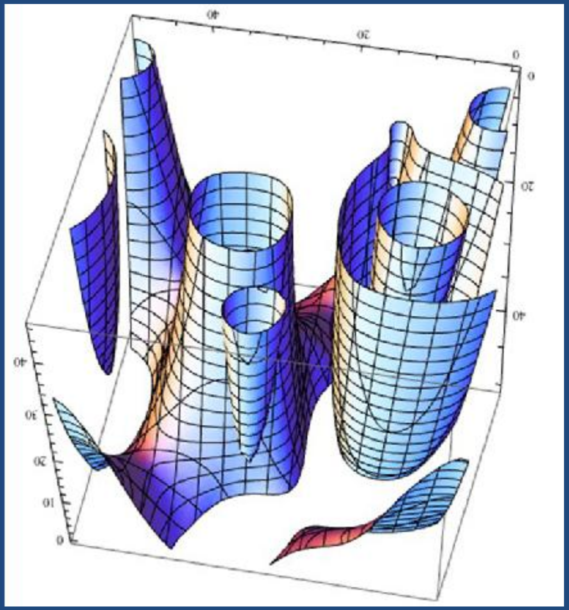
Bulk & boundary

Vorticity plays a key role. It is encoded everywhere!

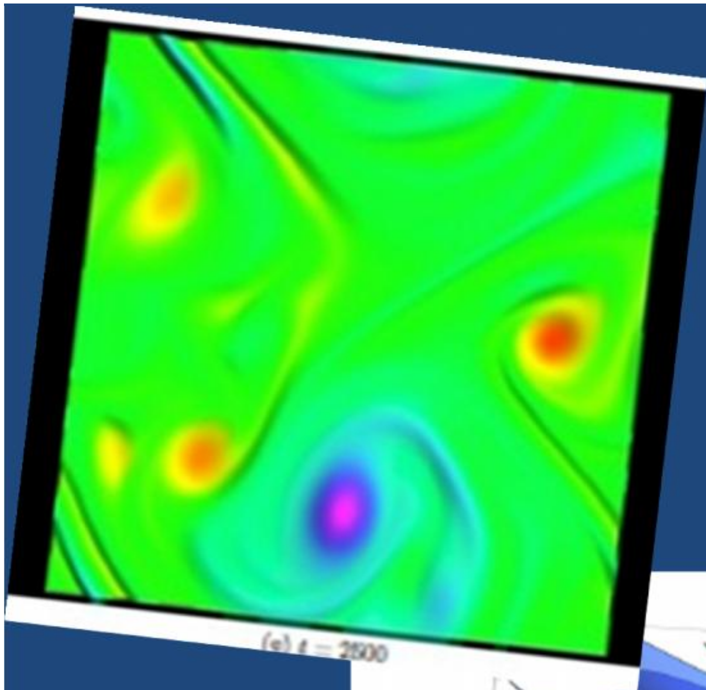
- (Adams-Chesler-Liu): Pontryagin density: $R_{abcd}\ {}^{*}R^{abcd} \sim \omega^2$
- (Eling-Oz): Im(Ψ_2) ~ T ω
- (Green,Carrasco,LL): *):* $Y_{1} \sim T^3 w$; $Y_{3} \sim T w$; $Y_{4} \sim i\, w/T$

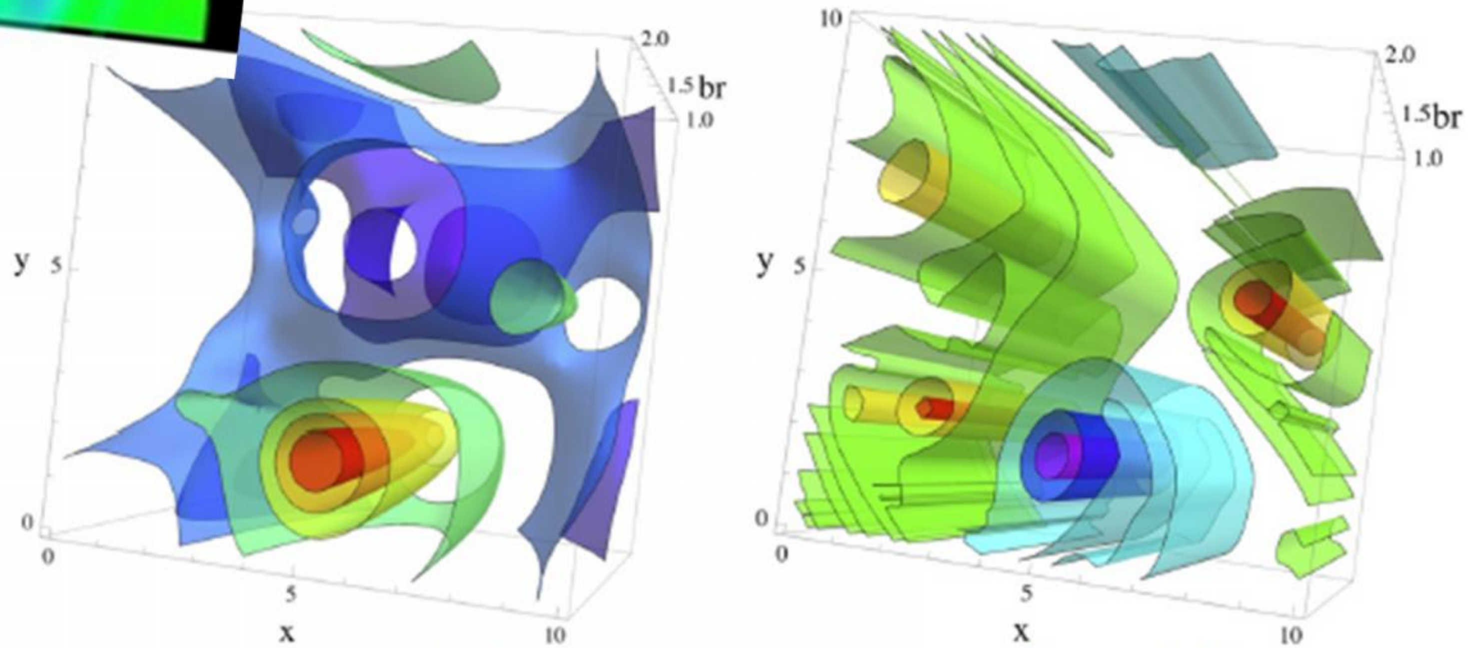
- Structure: (geon-like) gravitational wave ‘tornadoes’





From boundary to bulk

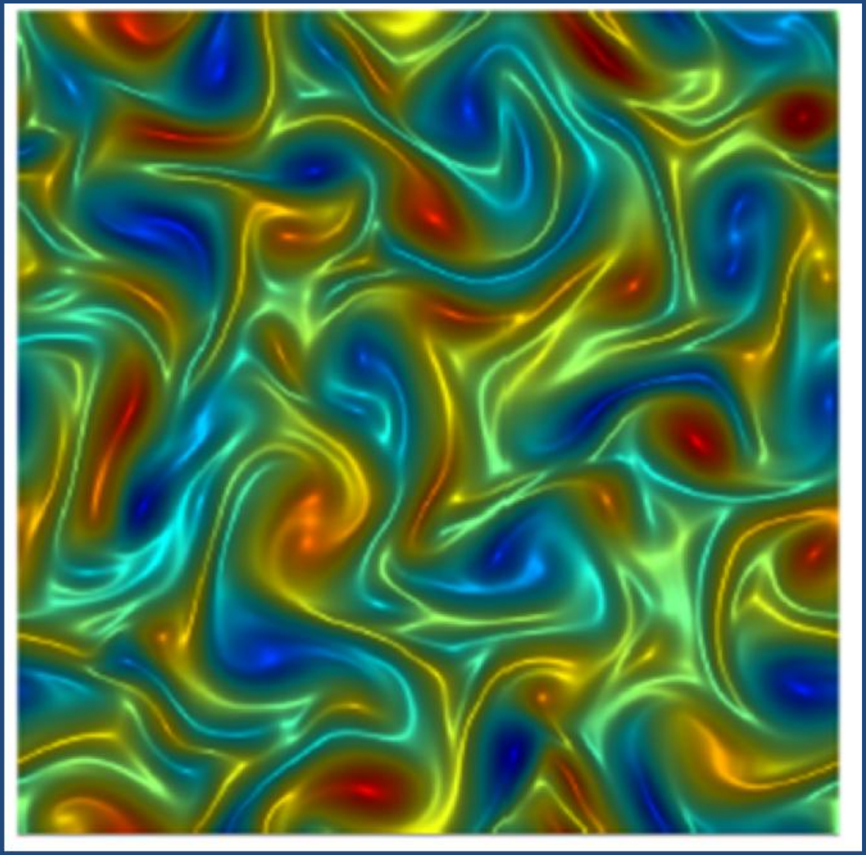




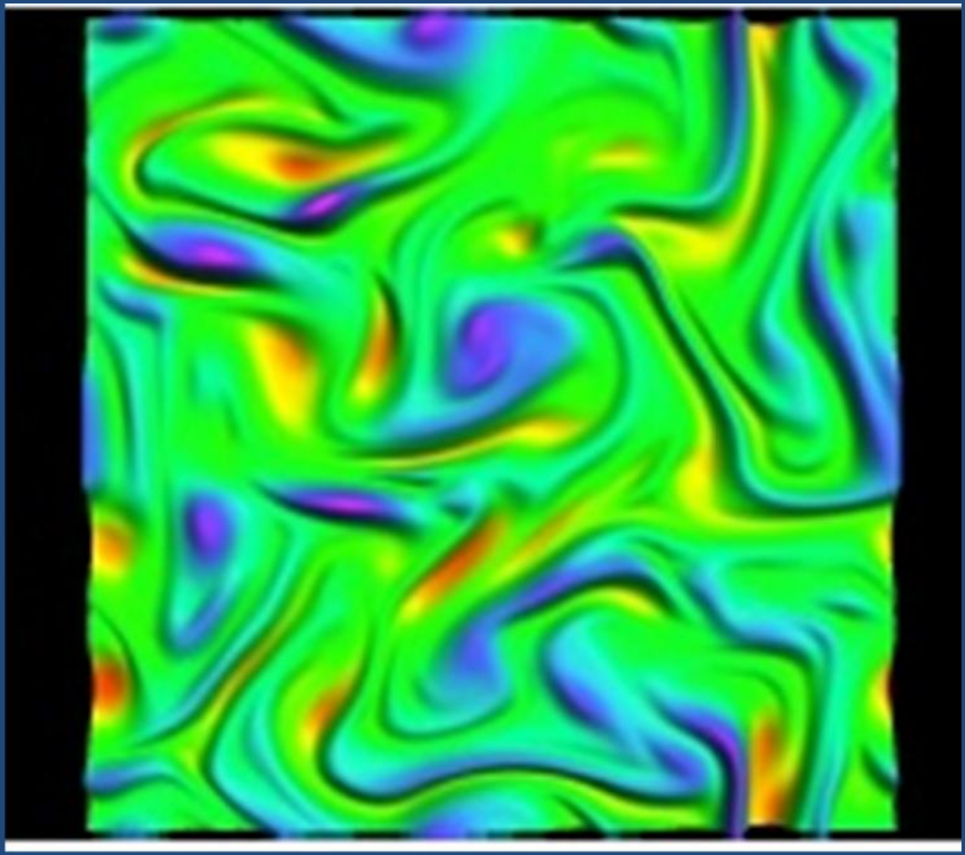
(a) $r^6\, C_{ABCD}C^{ABCD}$ (b) $r^7\, {}^*C_{ABCD}C^{ABCD}$

FIG. 8. Contour plots of principal invariants of the Weyl tensor in the bulk, computed from the zeroth order metric (2.1), from the simulation snapshot in Fig. 1e. Notice that (a) is representative of the energy density ρ, while (b) is representative of the vorticity, as expected from Eq. (B6).

Bulk & holographic calculation

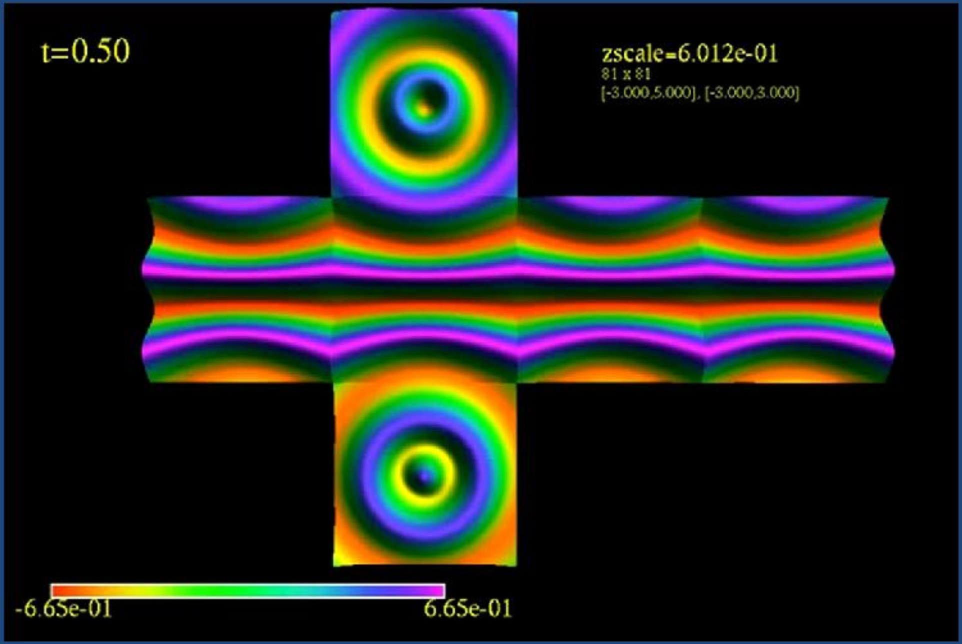


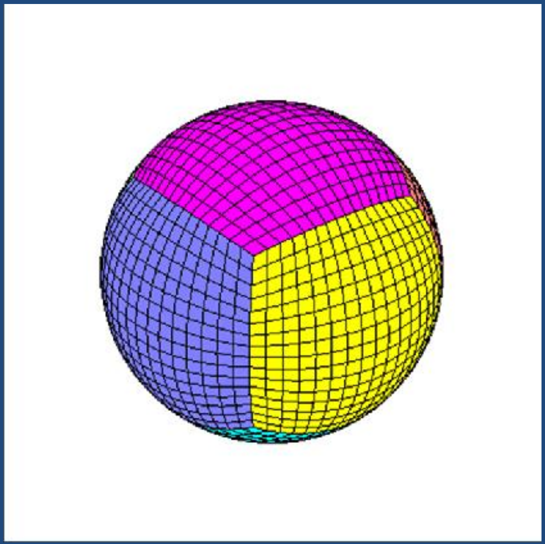
[Adams,Chesler,Liu]



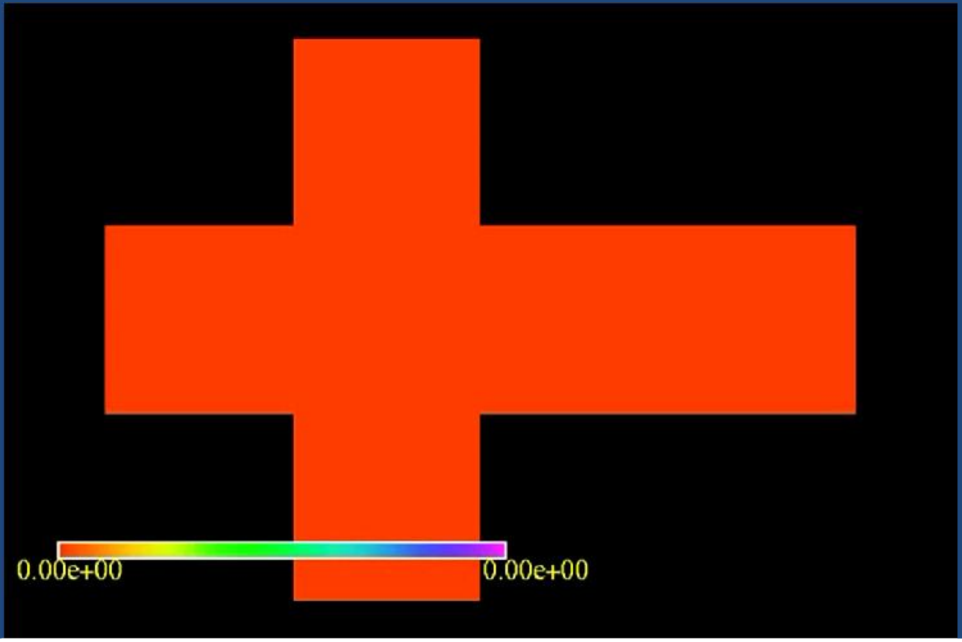
[Green,Carrasco,LL]

Global AdS

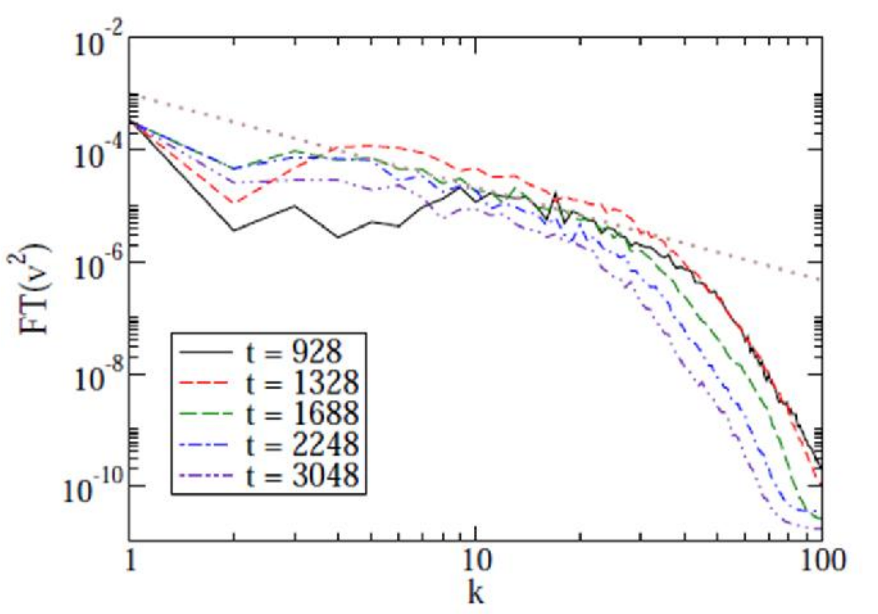


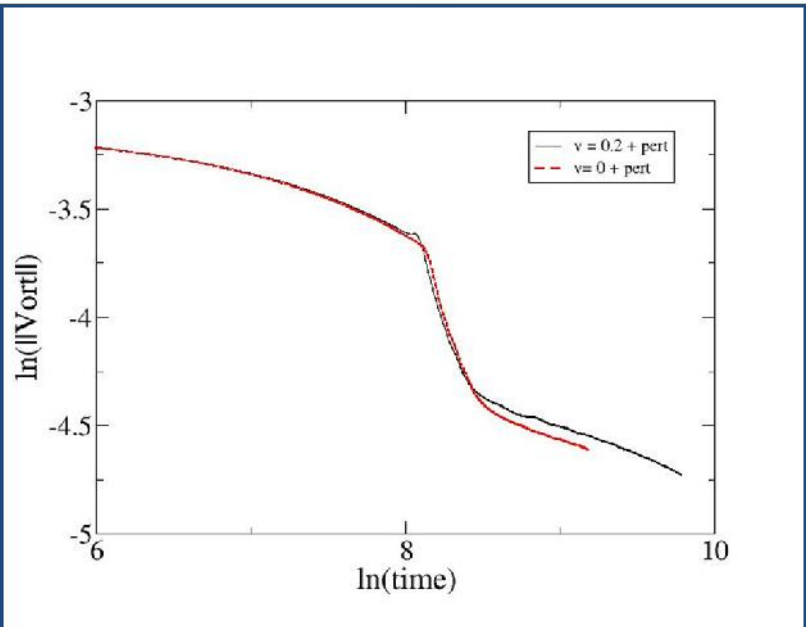


[we'll come back to this →]



---DECAYING TURBULENCE---
(warning : inertial regime? non-relativistic)



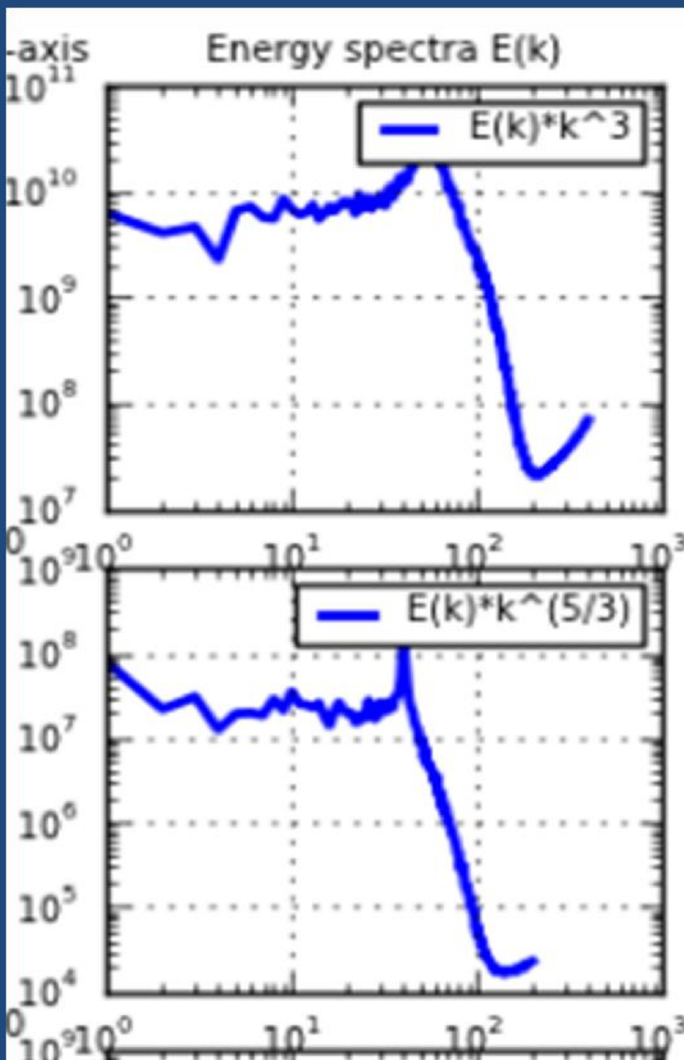


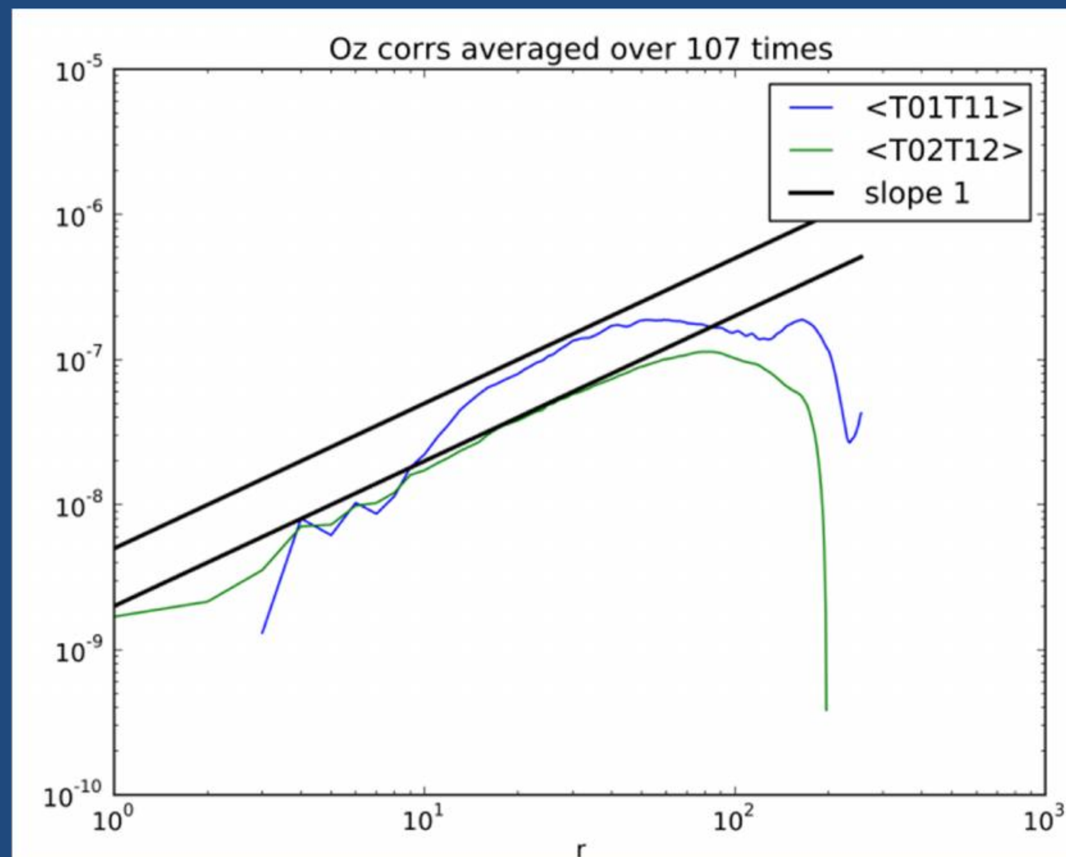
- Turbulence starts, eddies merge (if equal signs) into larger ones.
- Fully develop turbulence grows to ~ initial perturbation size
- Cascade to longer wavelengths
- Consitent with -5/3 Kolmogorov exponent
- Early/intermiedia/late stages → exponential/power-law/exponential decay
- Power law : ~ t^{-a} a ~ [0.5 – 1.5]
- $|w(t)|_2 \leq |w(0)|_1 t^{-0.5}$ [Thm: Gallay-Wayne '04]

--driven turbulence--

[ongoing!] ‘Fouxon-Oz’ scaling relation $<T_{0j}(0,t)\ T_{ij}(r,t)> = e\ r_i / d$

-must remove condensate [add friction or wavelet analysis]

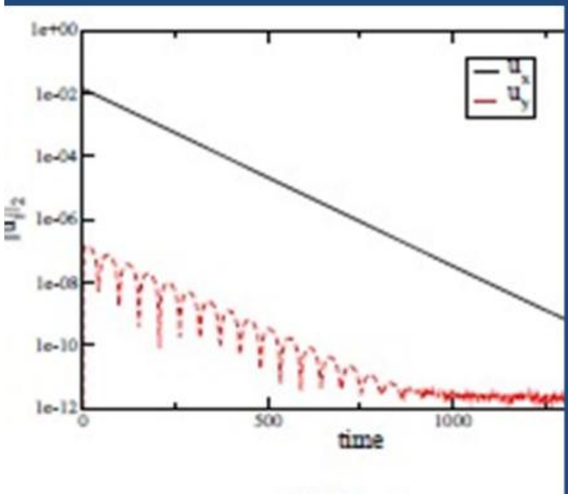




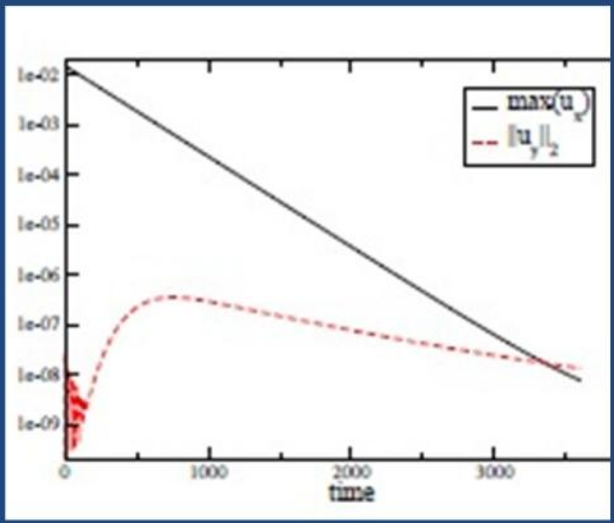
[Westernacher-Schneider,Green,LL]

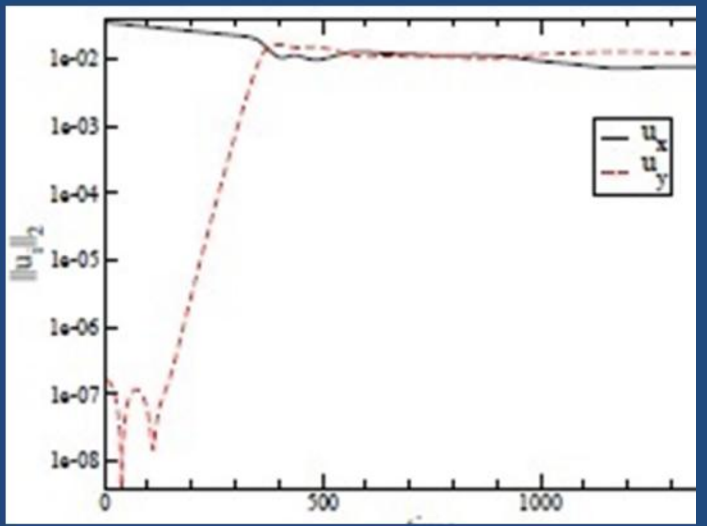
OK. Gravity goes turbulent in AdS. QNMs & Hydro: tension?

- QNMs aren't a basis [Warnick '13]
- Standard perturbation theory $Box_{g_{(0)}}(g_{(i)}) = S(g_{(i-1)})$, can't see an expn growth unless several orders are worked out or expansion is taken wrt time dependent backround <-> Laminar vs Turbulent behavior (transition can take place as Re is a function of time)



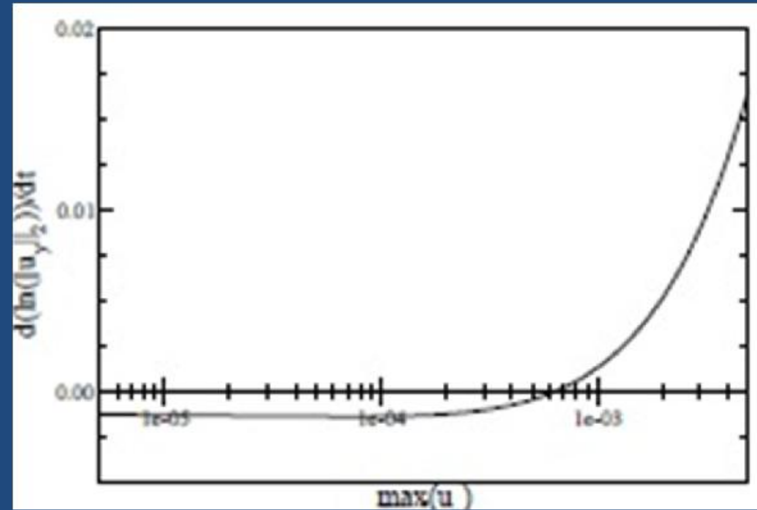
(b) Velocity





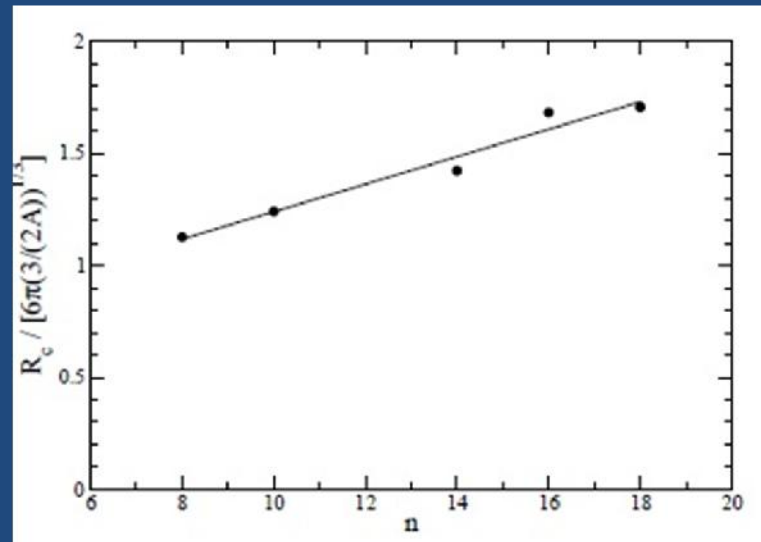
Reynolds number: R ~ ρ /ηγ λ

- Monitor when the mode that is to decay at liner level turns around with velocity perturbation. (R ~ v) →



- Monitor proportionality factor (R ~ λ) →

- Roughly R~ T L det(met_pert)



Can we model what goes on, and reconcile QNM intuition?...

- For a shear flow, with ρ = const. Equations look like ~

$$\frac{dx}{dt} + \alpha x = 0,$$
$$\frac{dy}{dt} + \beta y - \gamma x y = 0,$$

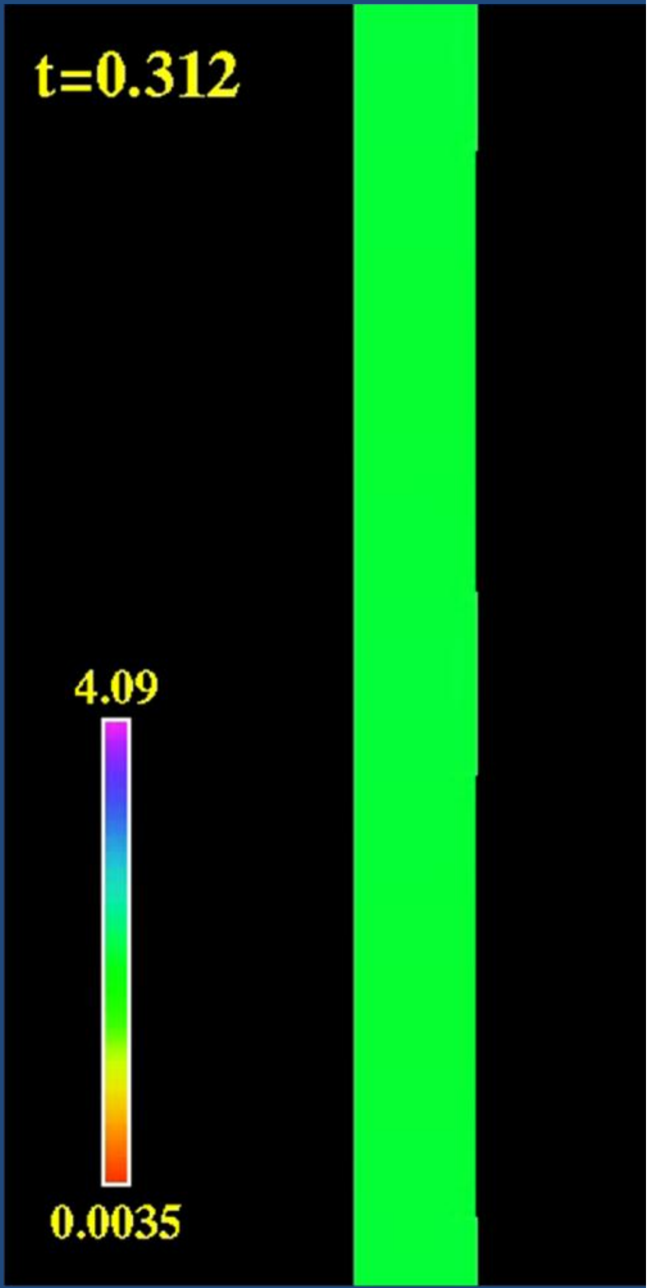
- Assume $x^{(0)} = 0$; $y^{(0)} = 0$.
- 'standard' perturbation analysis : to second order: exponential decaying solutions
- 'non-standard' perturbation analysis: take background as u0 + u1: → ie. time dependent background flow
  - Exponential growing behavior right away [TTF also gets it]

Observations

- Turbulence takes place in AdS – (effect varying depending on growth rate), and do so throughout the bulk (all the way to the EH)
  - Further, turbulence (in the inertial regime) is self-similar → fractal structure expected [Eling,Fouxon,Oz (NS case)]
  - *Assume Kolmogorov's scaling*: argue EH has a fractal dimension D=d+4/3 [Adams,Chelser, Liu. (relat case)]

- Aside: perturbed (unstable) black strings

induce fractal dim D=1.05 in 4+1 [LL,Pretorius]



More observations

- Inverse cascade carries over to relativistic hydro and so, gravity turbulence in 3+1 and 4+1 move in opposite directions [note, this is not related to Huygens' pple]
- Also…warning for GR-sims!, (the necessary) imposition of symmetries can eliminate relevant phenomena.
- Consequently 4+1 gravity equilibrates more rapidly (→ direct cascade dissipation at viscous scales which does not take place in 3+1 gravity) [regardless of QNM differences]
  - 2+1 hydro → if initially in the correspondence stays ok
  - 3+1 hydro → can stay within the correspondence (viscous scale!)

- From a hydro standpoint: geometrization of hydro in general and turbulence in particular:
  - Provides a new angle to the problem, might give rise to scalings/Reynolds numbers in relativistic case, etc. Answer long standing questions from a different direction*. However, to actually do this we need to understand things from a purely gravitational standpoint. E.g. :*
  - *What mediates vortices merging/splitting in 2 vs 3 spatial dims?*
  - *Can we interpret how turbulence arises within GR?*
  - *Can we predict global solns on hydro from geometry considerations? (e.g. Oz-Rabinovich '11)*

On to the 'real world'

- Ultimately what triggered turbulence?
  - AdS 'trapping energy' → slowly decaying QNMs & turbulence
  - Or slowly decaying QNMs → time for non-linearities to ``do something''?

- In AF spacetimes, claims of fluid-gravity as well. *However* this is delicate. Let's try something else, taking though a page from what we learnt from fluids.

- First, recall the behavior of parametric oscillators:
  - $q_{,tt} + \omega^2 (1 + f(t)) q + \gamma q_{,t} = 0$
  - Soln is generically bounded in time *except* when f(t) oscillates approximately with $\omega' \sim 2\omega$. [e.g. $f(t) = f_o \cos(\omega' t)$] . If so, an unbounded solution is triggered behaving as $e^{\alpha t}$ with $\alpha = (f_o^2 \omega^2/16 - (\omega'-\omega)^2)^{1/2} - \gamma$
  - (referred to as *parametric instability* in classical mechanics and optics)

[Yang-Zimmerman,LL]

Take a Kerr BH

- Let's consider now a BH with a mode that perturbs it with *(l,m)*
- Now, to linear order $g_{full} = g_{kerr} + h_1$ (h_1 -> $h_o(t) = \varepsilon\, e^{i\omega t}\, Y_{lm}$)
- QNMs → $\omega_{lmn} = {}^{m}/_{2} - {}^{\delta\sqrt{\kappa}}/_{\sqrt{2}} - i\,(n + {}^{1}/_{2})\ {}^{\sqrt{\kappa}}/_{\sqrt{2}}$
- with $\kappa = [1\text{-}a/m]^{1/2}$: thus,if sufficiently highly spinning, QNMs decay → 0.
- Consider the next order as determined by this –time dependent– background → parametric oscillator analogue!

- As a simplification: we consider a single mode for h_1 and we'll take only a scalar perturbation (the general case is similar). One obtains:

$$[\ \mathrm{Box}_{kerr} + O(h_1)\ ]\,\Phi = 0.$$

- With the solution having the form: $e^{t(\alpha - \omega_I)}$ with

$$\alpha = \pm\sqrt{|Hh_0(t)/Qm'|^2 - (\omega'_R - \omega_R/2)^2}\,,$$

- So exponentially growing solution if:

$$h_0(t)/(m'\omega'_I) - |Q/H|\sqrt{(\omega'_R - \omega_R/2)^2/\omega'^2_I + 1} > 0\,.$$

- → if Φ has l, m/2 → a parametric instability can turn on; i.e. inverse cascade.
- Further, one can find 'critical values' for growth onset.

(l,m)	$l'=1$	$l'=2$	$l'=3$	$l'=4$	$l'=5$	$l'=6$	$l'=7$	$l'=8$
$(2,2)$	0.287	0.163	0.130	0.122	0.117	0.115	0.113	0.111
$(4,2)$	43.2	62.1	92.7	123	118	118	117	117
$(4,4)$	–	3.62	0.00676	0.0114	0.0108	0.0104	0.0101	0.0100

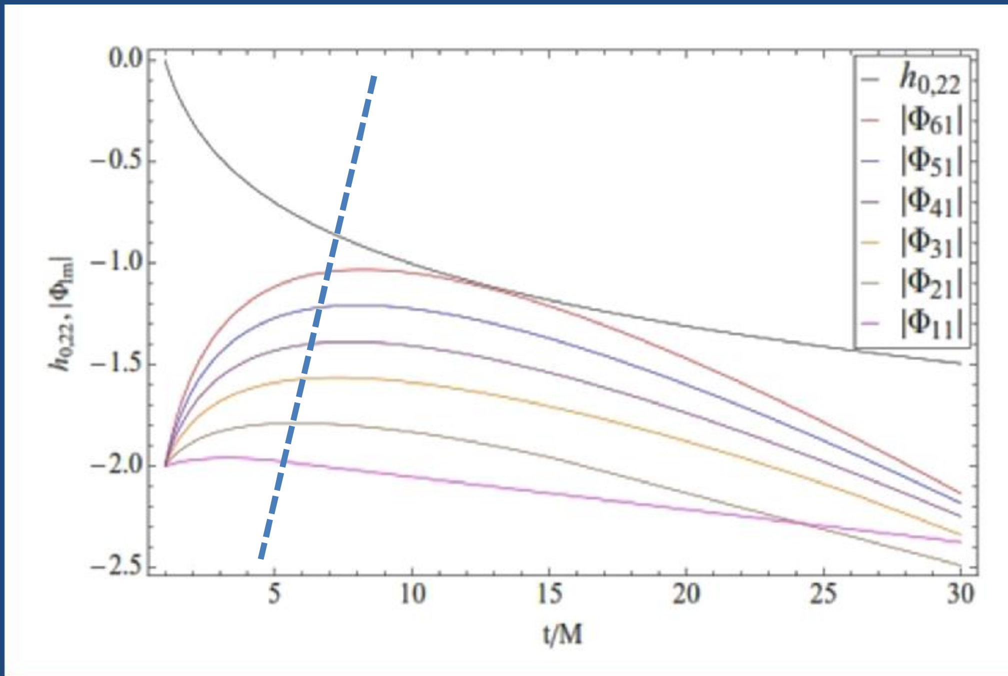
- And also one can define a max value as:

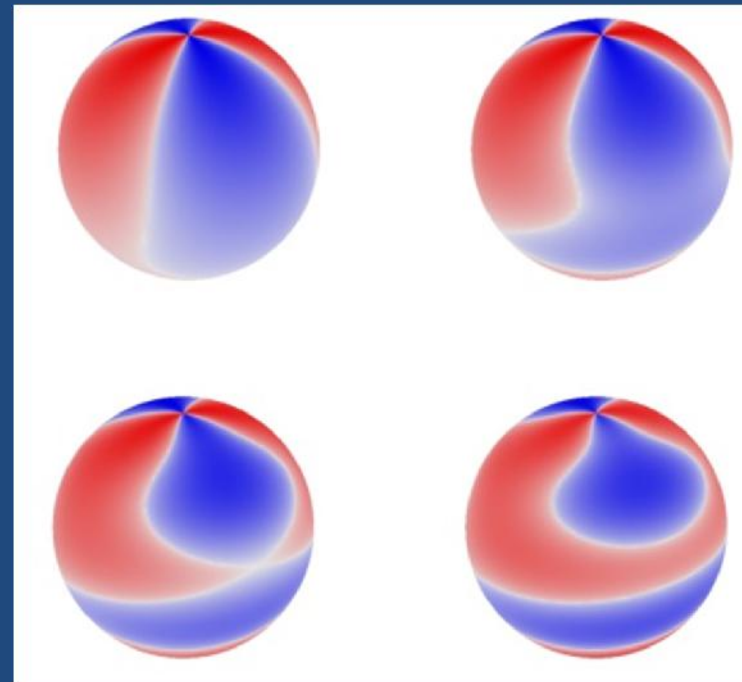
$$Re_g = h_o/(m\ \omega_\nu)$$

- identify λ <-> $1/m$; v <-> h_o ; ν /ρ<-> ω_ν

$$\rightarrow Re_g = Re$$

Critical ``Reynolds'' number & instability





a = 0.998, perturbation ~ 0.02%, initial mode l=2,m2

Could 'potentially' have observational consequences

Perhaps `obvious' from the Kerr/CFT correspondence ? (rigorous?)

more general?

Tantalizingly…. $h_o \sim \kappa^p$ [Hadar,Porfyaridis,Strominger], but also ω_ν → instability still possible!

$$h_0(t)/(m'\omega'_I) - |Q/H|\sqrt{(\omega'_R - \omega_R/2)^2/\omega'^2_I + 1} > 0 .$$

Final comments

Summary:

- Gravity does go turbulent in the right regime, and a gravitational analog of the Reynolds number can be defined
- AdS is ‘convenient’ but not necessary
- Some possible observable consequences
- ‘geometrization’ of turbulence is exciting/intriguing, what else lies ahead?

Some new chapters…

- 3+1 vs (>3)+1 gravitational perturbations and cascade? [anything to say as far a possible censor?]

- Fractal structure analysis & evolution?

$$-\Psi_{2,t} = -\partial\Psi_3 + \sigma\Psi_4 \rightsquigarrow \omega_{,t} = \partial\omega + F([\omega])$$

- Geometrical interpretation of 'hydro' quantities (e.g. ensthrophy, [Real-Freitas '14])

- Looking in "non-standard" places (e.g. cosmology [mixmaster behavior])