Stability Problems for Black Holes

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Two Problems

Problem 1: Prove *linear* and *non-linear* stability of Schwarzschild and Kerr.

Problem 2: Prove (in)stability of Kerr-AdS.

Of course, (in)stability of pure AdS is also open (previous talks).
I will report on recent progress concerning the above two problems.

The Strategy

Einstein's equations are wave equations: $\Box_g g_{\mu\nu} = \mathcal{N}(g, \partial g)$.
(Asymptotic) stability is based on **decay**.

1. Understand $\Box_g \psi = 0$ for g a black hole metric
2. Understand non-linear toy-problems: $\Box_g \psi = (\partial \psi)^2$
3. Reintroduce the tensorial nature
   (a) system of gravitational perturbations (linear)
   (b) full problem (non-linear)

Difficulties and Caveats

1. Decay needs to be sufficiently strong.
   The methods should be robust.
   $\rightarrow$ quantify the effect of ergosphere, trapped null-geodesics and the redshift on wave propagation.
2. Non-linearities need to have structure
3. (a) Gauge issues, stationary modes
   (b) mass and angular momentum of final state?

Problem 1

Results I: Quantitative decay of the linear scalar problem

Theorem 1. *[Dafermos–Rodnianski–Shlapentokh-Rothman 2005-2014] Solutions of the linear wave equation* $\Box_{g_{M,a}}\psi = 0$ *for* $g_{M,a}$ *a* *subextremal* *member of the Kerr-family decay* *polynomially in time* *on the black hole exterior.*

→ extremal case (Aretakis; Lucietti, Murata, Reall, Tanahashi)

Results II: Decay for non-linear toy-models

Theorem 2. *[J.Luk 2010] Small data solutions of the non-linear wave equation* $\Box_{g_{M,a}}\psi = \mathcal{N}_{nc}(\psi, \partial\psi)$ *for* $g_{M,a}$ *a Kerr spacetime with* $|a| \ll M$ *exist globally in time and decay polynomially in time on the black hole exterior.*

Results III: Linear Stability of Schwarzschild

Regarding item 3.(b) above, we were recently able to prove

Theorem 3. *[Dafermos–G.H.–Rodnianski] The Schwarzschild solution is linearly stable: Solutions to the system of gravitational perturbations decay to a linearised Kerr solution polynomially in time with quantitative decay rates and constants depending only on norms of the initial data.*

Linear stability of Kerr is completely open!

Mode stability is NOT linear stability!

- In the old days, people knew very well the difference between mode stability and linear stability, see for instance Whiting's 1989 paper "Mode Stability of the Kerr solution"
- Nowadays, one often sees Whiting being cited erroneously for proving linear stability of Kerr!

Mode stability excludes a particular type of exponentially growing solution. It does not rule out exponential growth in general let alone show that solutions are bounded or decay.

The latter would be linear stability.

Key Observations

1. linearization in double null-gauge
2. a quantity (combination of derivatives of curvature and connection coefficients) which
   (a) decouples from the system
   (b) satisfies a "good" wave equation (NOT Teukolsky)
   (c) controls all other dynamical quantities (hierarchy).

   $\rightarrow$ Chandrasekhar
3. all insights from the wave equation enter

What is left to do?

1. A non-linear problem which doesn't need Kerr: axisymmetric perturbations of Schwarzschild with $a = 0$
2. Generalise the linear stability result to Kerr
3. Do the full problem

Problem 2

Follow the strategy for AdS black holes...

Study $\Box_g \psi - \frac{\Lambda}{3}\alpha\psi = 0$ with $\alpha < \frac{9}{4}$ (Breitenlohner–Freedman).

Well-posedness non-trivial.

(Breitenlohner–Freedman, Bachelot, Ishibashi–Wald, GH, Vasy, Warnick)

Theorem 4. *[G.H.–Smulevici 2011–2013] Let $(\mathcal{R}, g)$ denote the exterior of a Kerr-AdS with parameters $M > 0$, $|a| < l$. Consider the massive wave equation (with Dirichlet boundary conditions)*

$$\Box_g \psi + \frac{\alpha}{l^2}\psi = 0 \qquad \textit{with } \alpha < 9/4$$

1. *The solutions arising from data prescribed on Σ_0 remain uniformly bounded, provided $r_{hoz}^2 > |a|l$ holds:*

$$\sum_{i=1}^{k} \int_{\Sigma_{t^\star}} |D^i\psi|^2 \leq C \sum_{i=1}^{k} \int_{\Sigma_0} |D^i\psi|^2 \qquad \textit{for } k \geq 1$$

2. *The solutions satisfy for $t^\star \geq 2$*

$$\int_{\Sigma_{t^\star}} |D\psi|^2 \leq \frac{C}{(\log t^\star)^2} \int_{\Sigma_0} |D\psi|^2 + |D^2\psi|^2$$

3. *The log-decay is sharp.*

Comments

1. The boundedness relies on the existence of a globally time-like Killing field in the non-superradiant regime (Hawking–Reall)
   Growing modes if HR is violated [Press-Teukolsky, Shlapentokh-Rothman]
2. Note the loss of derivatives (trapping) and the slow decay rate.
   This is due to a stable trapping phenomenon.
   In AdS-Schwarzschild any *fixed l* mode decays exponentially!
3. Construction of quasi-modes (see also Gannot)

Application: ultracompact neutron stars [Keir]
Generalisations: [G.H.–Warnick 2013] Boundedness for Neumann and some Robin boundary conditions.

Non-linear problems

The log-decay for the linear problem does not allow us to follow the usual strategy. $\rightarrow$ Too weak to prove small data global existence.

- Instability? [G.H-Smulevici]
- Stability? [Dias–Horowitz–Marolf–Santos]
- "Doable" problem: Construction of solutions converging to Kerr-AdS exponentially fast [cf. Dafermos–G.H.–Rodnianski 2013, Friedrich 1995]