Quantum Focussing and the Quantum Null Energy Condition

Raphael Bousso

Center for Theoretical Physics
University of California, Berkeley

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Quantum Information and Quantum Gravity

The study of Quantum Information is revolutionizing how we pursue Quantum Gravity.

As a byproduct, we are discovering new results in nongravitational physics (QFT), which can be (laboriously) proven.
Area Theorem for Event Horizons

Hawking 1971: In GR, the total area of all event horizons cannot decrease:

\[ dA \geq 0. \]
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\[ dA \geq 0 \, . \]

✓ Proven, assuming the Null Energy Condition (NEC):

\[ T_{kk} \equiv T_{\mu\nu}k^\mu k^\nu \geq 0 \, . \]

True for classical matter.
Black Hole Entropy

What happens to the entropy of a matter system that is lost into a black hole?
Cannot be accessed, not even by jumping after it.
2nd Law violated/transcended?
Black Hole Entropy

What happens to the entropy of a matter system that is lost into a black hole?
Cannot be accessed, not even by jumping after it.
2nd Law violated/transcended?

Bekenstein 1972: Black holes must themselves have entropy!

Inspired by area theorem:

$$S_{BH} \equiv \frac{A}{4G\hbar}$$

up to $O(1)$ factor.
Generalized Entropy

To an outside observer, the total entropy consists of the entropy of all the matter outside the black hole, plus the entropy of the black hole.

\[ S_{\text{gen}} \equiv \frac{A}{4G\hbar} + S_{\text{out}}. \]
Generalized Entropy

Side note: the sum

\[ S_{\text{gen}} \equiv S_{\text{out}} + \frac{A}{4G\hbar} + \text{subleading counterterms}, \]

is better defined than its constituents. The area is the most important in a set of local geometric counterterms that absorb divergences in the von Neumann entropy \( S_{\text{out}} \). 

Susskind & Uglum 1994, ...
Generalized Second Law for Event Horizons

In the presence of black holes, the ordinary 2nd law becomes the Generalized Second Law:

\[ dS_{\text{gen}} \geq 0. \]

Bekenstein 1972, 1973, 1974
Hawking Radiation

- Had to be there since $T^{-1} = dS/dE$.
- Found by explicit calculation. Hawking (1974)
- Black holes are thermodynamic objects.

The area decreases as they evaporate!
This is possible because the Null Energy Condition is violated. (Also, e.g., in Casimir energy.)

Amazingly, the Generalized Second Law still holds.

Hawking radiation increases $S_{\text{out}}$ enough to compensate for area loss.
The Facts So Far

The GSL is simultaneously a statement about geometry and about quantum info!

It becomes Hawking's area theorem in the classical limit.

It becomes the ordinary second law in the case where there are no black holes.

But neither law survives on its own, if black holes are present and treated at the quantum level.
Alternative Fact

From the New York *Alternative Times*, December 12, 1974:

**Stephen Hawking Discovers “2nd Law of Thermodynamics”**

Claims It Follows From General Relativity
Generalized Entropy as a Discovery Tool

- Start with a classical gravity theorem involving area. (Here, \( dA \geq 0 \)).
- Add a quantum correction to make it robust against violations of the Null Energy Condition:
  \[
  A \rightarrow A + 4G\hbar S_{\text{out}}.
  \]
- Take a limit where gravity becomes unimportant (here, \( A \rightarrow 0 \)).
- Obtain a quantum law. (Here, \( dS \geq 0 \)).
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Can we actually do this, starting with other GR theorems? Yes: Classical Focussing Theorem \(\rightarrow\) Quantum Focussing
Generalized Entropy as a Discovery Tool

Generalized entropy can be defined not just for slices of event horizons…

…but for any 2D surface $\sigma$ that divides space into two sides.

This means we can consider what GR tells us about general surfaces.

Let’s see what happens when we add a quantum correction, $A \rightarrow A + 4G\hbar S_{out}$ to appropriate GR formulas.
Expansion of Light-rays

The classical expansion, $\theta$, is the (logarithmic) derivative of an area element, when transported along orthogonal light-rays.
In General Relativity, matter focusses light: 

\[ \theta' \leq 0 \, . \]

Like the Area Theorem, this assumes the Null Energy Condition, \( T_{kk} \geq 0 \). Quantum effects can violate this. Example: evaporating black hole.

→ Formulate a more robust, quantum-corrected focussing theorem!
Define a quantum expansion using \( A \rightarrow S_{\text{gen}} \equiv A + 4G\hbar S_{\text{out}} \):

\[
\Theta[\sigma; y_1] \text{ is the rate (per unit area) at which the generalized entropy changes when an infinitesimal area element of } \sigma \text{ at } y_1 \text{ is deformed in one of its future orthogonal null directions.}
\]

RB, Fisher, Leichenauer & Wall, 2015
Classical Expansion $\rightarrow$ Quantum Expansion

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RB, Fisher, Leichenauer & Wall, 2015
Classical Expansion → Quantum Expansion

Define a quantum expansion using $A \rightarrow S_{\text{gen}} \equiv A + 4G\hbar S_{\text{out}}$:

$$\Theta = \theta + \frac{4G\hbar}{A} S'_{\text{out}}.$$ 

$\Theta[\sigma; y_1]$ is the rate (per unit area) at which the generalized entropy changes when an infinitesimal area element of $\sigma$ at $y_1$ is deformed in one of its future orthogonal null directions.  

RB, Fisher, Leichenauer & Wall, 2015
Classical $\rightarrow$ Quantum Focussing Conjecture

The classical expansion will not increase along any light-ray,

$\theta' \leq 0$,

if............................... the NEC holds.
Classical $\rightarrow$ Quantum Focussing Conjecture

The quantum expansion will not increase along any light-ray,
\[ \Theta' \leq 0 , \]
regardless of whether the NEC holds.

RB, Fisher, Leichenauer & Wall, 2015
GR $\rightarrow$ QG $\rightarrow$ QFT

We lifted a GR theorem to a (semi-classical) quantum gravity conjecture,

$$\Theta' \leq 0.$$ 

One can check that the QFC does, indeed, hold for an evaporating black hole. It has many interesting implications, some new, some previously known. I will describe some of these later, if time allows.
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But first, let’s throw away* the gravity part, and learn something new about QFT! We obtain the Quantum Null Energy Condition.
From the QFC to the QNEC

\[ \Theta = \theta + \frac{4G\hbar}{A}S_{\text{out}}' \cdot \]

Expanding \( \Theta \) into its classical and quantum part, we notice that the first term generally dominates, because it is \( O(G^0) \).

For example, if the initial surface is a sphere in Minkowski space, \( \theta = 2/R \).
We can suppress such geometric contributions to $\theta$, if we choose the initial surface to be a flat plane in Minkowski space. Then initially $\theta = 0$. 
From the QFC to the QNEC

\[ \Theta = \theta + \frac{4G\hbar}{A} S'_{\text{out}}. \]

Away from the initial surface, \( \theta \) will not vanish, because gravity bends the light-rays.

But now, the leading contributions to \( \theta \) are \( O(G) \), and so are of the same order as the “quantum correction.”
From the QFC to the QNEC

\[ \Theta = \theta + \frac{4G\hbar}{A} S'_{\text{out}} \cdot \]

Finally, we can “tame” the effects of gravity by taking \( G \to 0 \). This ensures that only the \( O(G) \) term contributes to \( \Theta \).

It also means that \( G \) cancels out as an overall factor. This is why the final result makes no reference to gravity at all. It is a QFT statement.
From the QFC to the QNEC

The QFC becomes

$$0 \geq \Theta' = \theta' + \frac{4G\hbar}{\mathcal{A}} (S''_\text{out} - S'_\text{out}\theta)$$

$$= -\frac{1}{2} \theta^2 - \zeta^2 - 8\pi G \langle T_{kk} \rangle + \frac{4G\hbar}{\mathcal{A}} (S''_\text{out} - S'_\text{out}\theta)$$

For a null surface with vanishing classical shear and expansion, $\theta = \zeta = 0$, this implies

$$\langle T_{kk} \rangle \geq \frac{\hbar}{2\pi} \lim_{\mathcal{A} \to 0} \frac{S''_\text{out}}{\mathcal{A}} \quad (QNEC).$$
Quantum Null Energy Condition

\[ \langle T_{kk} \rangle \geq \frac{\hbar}{2\pi} \lim_{\mathcal{A} \to 0} \frac{S''_{\text{out}}}{\mathcal{A}}. \]

First lower bound on the local energy density.

RHS: nonlocal, information-theoretic quantity.

Conversely, the local energy density limits how rapidly one can increase the rate at which information is acquired.
Quantum Null Energy Condition

\[ \langle T_{kk} \rangle \geq \frac{\hbar}{2\pi} \lim_{A \to 0} \frac{S_{\text{out}}''}{A}. \]

Since $G$ dropped out, we can try to prove this statement within QFT.
Proof for Free Fields

4.1 The Replica Trick

The replica trick prescription is to use the following formula for the von Neumann entropy [60]:

\[ S_{n} = - \text{Tr} \rho \log \rho = (1 - n\lambda_{0}) \log \text{Tr} \rho^{n} \bigg|_{n=1}, \]  

(4.1)

This can be written as

\[ S_{n} = \mathcal{D} \log \bar{Z}_{n}, \]  

(4.2)

where \( \bar{Z}_{n} = \text{Tr} \rho^{n} \) and the operator \( \mathcal{D} \) is defined by

\[ \mathcal{D} f(n) = (1 - n\lambda_{0}) f(n) \bigg|_{n=1}, \]  

(4.3)

where \( f(n) \) is some function of \( n \). Since \( \bar{Z}_{n} \) is only defined for integer values of \( n \), we first must analytically continue to real \( n > 0 \) in order to apply the \( \mathcal{D} \) operator. The analytic continuation step is in general quite tricky, and will require care in our calculation. (Our analytic continuation is performed in Section 4.4.)

On general grounds discussed above, we must study the second-order term in a perturbative expansion of the entropy about the state \( \rho^{(0)} \). Suppressing all \( \lambda \) dependence, we have

\[ \bar{Z}_{n} = \text{Tr} \left( \rho^{(0)} + \sigma \right)^{n}, \]  

(4.4)

Expanding \( \bar{Z}_{n} \) to quadratic order to include \( \mathcal{D} \sigma^{2} \), we have

\[ \bar{Z}_{n} = \text{Tr} \left( \rho^{(0)} \right)^{n} + n \text{Tr} \left[ \sigma \rho^{(0)} \right]^{n-1} + \frac{n(n-1)}{2} \text{Tr} \left( \left[ \rho^{(0)} \right]^{n} \sigma^{2} \right) + \cdots. \]  

(4.5)

Using the notation introduced in (3.10) we can write

\[ \bar{Z}_{n} = \text{Tr} \left( \rho^{(0)} \right)^{n} + n \text{Tr} \left[ \mathcal{O} \rho^{(0)} \right]^{n-1} + \frac{n(n-1)}{2} \text{Tr} \left[ \mathcal{O} \rho^{(0)} \right]^{n-1} \mathcal{O}^{2} \rho^{(0)} + \cdots. \]  

(4.6)

We denote by \( \mathcal{O}^{(2)} \) the operator \( \mathcal{O} \) conjugated by \( \rho^{(0)^{2}} \):

\[ \mathcal{O}^{(2)} \equiv \rho^{(0)^{2}} \mathcal{O} \rho^{(0)^{2}}. \]  

(4.7)

In the replica trick one often works with the partition function \( Z_{n} \), in terms of which \( \bar{Z}_{n} = Z_{n} / (2^{n}). \) Choosing \( Z_{n} \) over \( \bar{Z}_{n} \) is equivalent to choosing a different normalization for \( \rho \), but we find it convenient to keep \( \lambda_{0} = 1 \).
Proof for Free Fields

Using (3.11), one can write the sum over replicas in (4.26) as follows:

\[ \left( \sum_{i=0}^{j-1} \mathcal{O}^{(i)} \right)_n = \left( \sum_{i=0}^{j-1} \int dx \, dy \, f_i(r, \delta \theta) \phi \partial \phi \right)_n \]  

(4.25)

This equality comes from interpreting \( \mathcal{O}^{(i)} \) as \( \mathcal{O} \) inserted on the \((k+1)\)th replica sheet (see (4.7)). Summing over sheets and integrating \( \theta \in [0, 2\pi] \) on each one is equivalent to just integrating \( \theta \in [0, 2\pi] \), which gives the entire replicated manifold. The definition of \( \mathcal{O} \) for angles greater than \( 2\pi \) is given by the the Heisenberg evolution rule, the right hand side of (3.5). The field is still holomorphic, but it would be misleading to write it as a function of \( \mathbf{r} \) since it is not periodic in \( \theta \) with period \( 2\pi \).

Because the \( \phi_i(r, \theta) \) are not dynamical, they should be identical on each sheet. In the Fourier representation as in (4.21), this means keeping the Fourier coefficients fixed and keeping the mass parameters intact. Thus we have

\[ \left( \sum_{i=0}^{j-1} \mathcal{O}^{(i)} \right)_n = \frac{1}{2(2\pi)^2} \sum_{i=0}^{j-1} \int dx \, dy \, f_i(r, \delta \theta) \phi \partial \phi \]  

(4.26)

The CFT two point function is calculated in Appendix A.1:

\[ \langle \phi \partial \phi \langle \phi \partial \phi \rangle \rangle_n = \frac{1}{1 + \text{sgn}(\mathbf{r} \cdot \mathbf{r} - 1)} \left( \frac{\mathbf{r}}{\mathbf{r}^2} \right)^n \]  

(4.27)

\[ \langle \phi \partial \phi \langle \phi \partial \phi \rangle \rangle_n = \frac{1}{1 + \text{sgn}(\mathbf{r} \cdot \mathbf{r} - 1)} \left( \frac{\mathbf{r}}{\mathbf{r}^2} \right)^n \]  

(4.28)

where \( \phi \) takes values in the integers divided by \( n \), and

\[ P(q, \mathbf{r}, \mathbf{r}') = \phi^q - 1 \left( \frac{\mathbf{r}}{\mathbf{r}^2} \right)^n \]  

(4.29)

When \( n = 1 \) there are no nonzero terms in the sum, but when \( n > 1 \) the answer is nonzero. For future convenience, we separate the parts which depend on \( \delta \) from those that do not.

The auxiliary system two point function is calculated in Appendix A.2:

\[ \langle \phi \partial \phi \langle \phi \partial \phi \rangle \rangle_n = \frac{1}{1 + \text{sgn}(\mathbf{r} \cdot \mathbf{r} - 1)} \left( \frac{\mathbf{r}}{\mathbf{r}^2} \right)^n \]  

(4.30)

where \( p \) is also an integer divided by \( n \) and \( P_{n}(q) = \frac{1}{1 + \text{sgn}(q - 1)} \) is a normalisation factor. Substituting this equation as well as (4.26) into (4.25) gives

\[ \mathcal{D} = \frac{1}{2(2\pi)^2} \sum_{i=0}^{j-1} \int dx \, dy \, f_i(r, \delta \theta) \phi \partial \phi \]  

(4.31)

\[ \left( \sum_{i=0}^{j-1} \mathcal{O}^{(i)} \right)_n = \frac{1}{2(2\pi)^2} \sum_{i=0}^{j-1} \int dx \, dy \, f_i(r, \delta \theta) \phi \partial \phi \]  

(4.32)

In going to the last line, we used the fact that the sum in brackets vanishes when \( n = 1 \) and that, for any two functions \( f(n), g(n) \) such that \( f(1) \) and \( \sum f(n) n \) are finite and \( g(1) = 0 \), the following relation holds:

\[ \mathcal{D} \langle \phi \partial \phi \langle \phi \partial \phi \rangle \rangle_n = \sum_{n=1}^{\infty} f(n) \mathcal{D} \langle \phi \partial \phi \langle \phi \partial \phi \rangle \rangle_n \]  

(4.33)

We now turn to the analytic continuation and application of \( \mathcal{D} \) on the term in brackets in (4.32). We will take care of the awkward \( \text{sgn}(\mathbf{r} \cdot \mathbf{r} - 1) \) part of the sum as two sums with positive argument. We will suppress the \( (r, r') \) dependence for the rest of the calculation:

\[ \sum_{q=1}^{\infty} \frac{\text{sgn}(P(q))}{q + m + n \mathbf{a}_q} = \sum_{q=1}^{\infty} \frac{P(q)}{q + m + n \mathbf{a}_q} - \frac{P(-q)}{q - m - n \mathbf{a}_q} \]  

(4.34)

Now we write \( q = k/n \) to turn this into a sum over integers:

\[ \sum_{q=1}^{\infty} \frac{P(q)}{q + m + n \mathbf{a}_q} + \frac{P(-q)}{q - m - n \mathbf{a}_q} \]  

(4.35)

In the next section we will see how to evaluate and analytically continue such sums quite generally.
Proof for Free Fields

- applies to free or superrenormalizable bosonic fields, stationary null surfaces
- null quantization → operator algebra factorizes over generators ("pencils")
- each pencil is 1+1 CFT
- in any global finite energy state, individual pencils are near the vacuum → small expansion parameter

RB, Fisher, Koeller, Leichenauer & Wall, 2015
Proof for Free Fields

- expand state, $\rho = \rho_0 + \sigma(\lambda)$
- expand entropy in powers of $\sigma$, $S = \sum S^{(i)}$
- find that $(S^{(0)} + S^{(1)})''$ would saturate the QNEC
- compute $S^{(2)''}$ using replica trick, prove $< 0$. 

RB, Fisher, Koeller, Leichenauer & Wall, 2015
Proof for Interacting Theories with Gravity Dual

Interacting theories with a gravity dual satisfy the QNEC.  

Koeller & Leichenauer, 2015

This follows

- from entanglement wedge nesting (Wall 2012),
- which in turn is necessary for consistent subregion duality.
Extension to Higher Curvature Gravity

Explored by Fu, Koeller, Marolf (2017a)

Apparent counterexample resolved by Leichenauer.
Extension to Curved Space

Explored by Fu, Koeller, Marolf (2017b)

Soon after, resolved by Akers, Chandrasekharan, Leichenauer, Levine, Shahbazi Moghaddam: the same conditions on the geometry are necessary and sufficient for all three of the following statements:

- $\text{QFC} \implies \text{QNEC}$ for free theories
- $\text{AdS/CFT } + \text{RT} \implies \text{QNEC}$ for theories with gravity dual
- $\text{QNEC}$ well-defined
Faulkner’s General Proof

A General Proof of the Quantum Null Energy Condition

Srivatsan Balakrishnan, Thomas Faulkner, Zuhair U. Khandker, Huajia Wang

Department of Physics, University of Illinois, 1110 W. Green St., Urbana IL 61801-3080, U.S.A.

Abstract

We prove a conjectured lower bound on \( \langle T_{--}(x) \rangle_\psi \) in any state \( \psi \) of a relativistic QFT dubbed the Quantum Null Energy Condition (QNEC). The bound is given by the second order shape deformation, in the null direction, of the geometric entanglement entropy of an entangling cut passing through \( x \). Our proof involves a combination of the two independent methods that were used recently to prove the weaker Averaged Null Energy Condition (ANEC). In particular the properties of modular Hamiltonians under shape deformations for the state \( \psi \) play an important role, as do causality considerations. We study the two point function of a “probe” operator \( \mathcal{O} \) in the state \( \psi \) and use a lightcone limit to evaluate this correlator. Instead of causality in time we consider causality in modular time for the modular evolved probe operators, which we constrain using Tomita-Takesaki theory as well as certain generalizations pertaining to the theory of modular inclusions. The QNEC follows from very similar considerations to the derivation of the chaos bound and the causality sum rule. We use a kind of defect Operator Product Expansion to apply the replica trick to these modular flow computations, and the displacement operator plays an important role. Our approach was inspired by the AdS/CFT proof of the QNEC which follows from properties of the Ryu-Takayanagi (RT) surface near the boundary of AdS, combined with the requirement of entanglement wedge nesting. Our methods were, as such, designed as a precise probe of the RT surface close to the boundary of a putative gravitational/stringy dual of any QFT with an interacting UV fixed point. We also prove a higher spin version of the QNEC.
Faulkner’s General Proof

Combines techniques used in two recent proofs of the ANEC (from quantum info/from causality).
QFC Implies the Covariant Entropy Bound

Consider the case where the generalized entropy is initially decreasing away from the surface $\sigma$.

Then the QFC implies that $S_{\text{gen}}$ cannot increase anywhere along $N$, and hence

$$S_{\text{gen}}[\sigma'] \leq S_{\text{gen}}[\sigma].$$
QFC Implies the Covariant Entropy Bound

Using $S_{\text{gen}} = S_{\text{out}} + A/4G\hbar$, we obtain:

$$\frac{A[\sigma] - A[\sigma']}{4G\hbar} \geq S_{\text{out}}[\sigma'] - S_{\text{out}}[\sigma].$$

For isolated matter systems on $N$, and in the special case where $A[\sigma'] = 0$, we recover the Covariant Entropy Bound, $S(N) \leq A/4G\hbar$. (“The world is a hologram.”)
Area Theorem for Holographic Screens

A future holographic screen is a 2+1D hypersurface foliated by marginally trapped 2-surfaces $\sigma(r)$.
Area Theorem for Holographic Screens

Assume Einstein’s equations and the NEC. Then

\[ \frac{dA}{dr} > 0 . \]

The area of a past or future holographic screen increases monotonically along its (unique) foliation.

✓ Proven, assuming the Null Energy Condition.

RB & Engelhardt, 2015a,b
Definition: A future (past) Q-screen is a hypersurface foliated by marginally quantum (anti-)trapped surfaces.

Conjecture: A past or future Q-Screen obeys the GSL:

$$dS_{\text{gen}} \geq 0.$$
2nd Law for Cosmology

The cosmological 2nd law, too, is implied by the Quantum Focussing Conjecture.

RB & Engelhardt, 2015c
Quantum Focussing Conjecture

The QFC appears to be quite powerful. It implies as special cases:
- classical focussing theorem
- Bekenstein’s GSL (and so Hawking’s area theorem) for black holes
- Covariant Entropy Bound
- Quantum Null Energy Condition
- a new GSL for cosmology (and a new area theorem)
These are just a few examples of the powerful interplay between quantum information and spacetime geometry.

See also: AdS/CFT; computational complexity; information paradox; firewall paradox;...