When to try Numerical Relativity

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When *not* to try Numerical Relativity

In GR we seek solutions to the field equations,

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab} \, .$$

Describe how curvature of spacetime evolves and is related to matter distribution. Non-linear PDEs. How can we find solutions? Different physics:

- Is the desired solution time independent, or with a lot of symmetry? There might be an exact solution. Read a book.
- Is the desired solution close to an exact solution that we already know? Perturbation theory. Allow for bigger perturbations by going to second-order. (Leor B).
- Compact objects well-separated and moving slowly? Use Newtonian or post-Newtonian expansion to construct accurate approximate solutions. (Laura B).

When to try Numerical Relativity

On the other hand:

- Are the fields highly dynamical? Are non-linear terms expected to dominate the behavior? Is there no symmetry? Use Numerical Relativity; numerical methods to find approximate solutions that converge to the continuum solution as you use more computational resources.

Ultimately I want you to think of NR as just one of the ways to study gravity; the only one that can reach some solutions. It is really a part of computational physics, although for historical reasons, regrettably parts of NR have developed independently.

A brief history of Numerical Relativity

From Wikipedia NR page and Smarr:

- Late 1950s: ADM decomposition.
- 1960s. First numerics, including Smarr [nice talk: http://online.itp.ucsb.edu/online/numrel00/smarr/].
- 1970s. Axisymmetric spacetimes. York 3+1 decomposition.
- 1980s. NR a field.
- 1990s. Critical Phenomena. 3d work becomes feasible. Grand challenge.
- 2000s. The compact binary *breakthrough*.
- 2010s. Model building. Detection of GWs! (Chris VDB).

An overview for how to do Numerical Relativity successfully

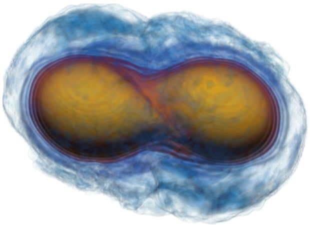
In this, and the next 7 slides I want to give a recipe of what I consider the “right” way to do NR. The seven points are:

1. Identify physical problem.
2. Formulation.
3. PDEs analysis.
4. Select numerical method.
5. Implementation. Extensive error testing.
6. Evaluate errors.
7. Physical interpretation.

A recipe for Numerical Relativity 1

1. Identify physical problem to solve. Obviously this should not belong to the class identified in the first slide. Examples include:

- Principle: Collapse of scalar or gravitational wave.
- GW astronomy: Compact binaries. 'Most important' current problem.
- Astrophysics: Supernova core collapse.



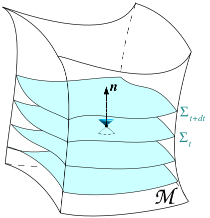
SXS BNS evolution. Vincent et. al. 2019. (Carlos P).

Must understand limits described by other methods. Otherwise if something weird is happening, you'll have *no idea*.

A recipe for Numerical Relativity 2

2. Formulation of the field equations. There is great freedom in writing down the field equations. Different methods:

- Constraints: conformal decomposition.
- Free-evolution.
- Constrained evolution.
- Characteristic formulation.



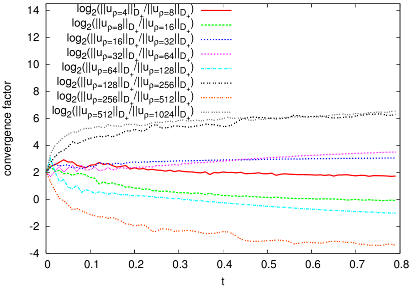
Brilliant notes: Gourgoulhon 2007.

To minimize computational cost, we should also choose appropriate coordinates, symmetry reduction, and evolved quantities. Basically geometric consideration.

A recipe for Numerical Relativity 3

3. PDEs analysis of the formulation. The name says it all. Remember, you *must* adjust formulation if necessary. “necessary”?

- Well-posedness: Unique solution, depending continuously on given data.
- If not PDE unsatisfactory; no numerical approach can work. What happens?
- Sounds like GR is broken. Really a feature of gauge freedom.



DH & Cao: 2011.

Going beyond short-term existence becomes *very* mathematical. One place where NR meets mathematical relativity.

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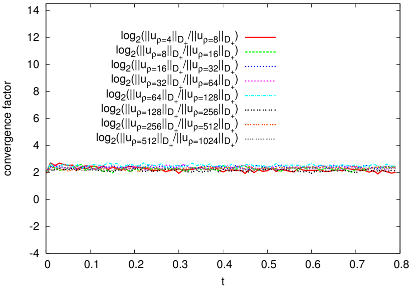

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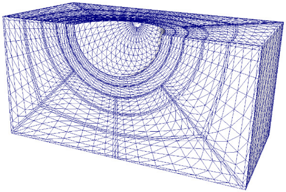
A recipe for Numerical Relativity 4

4. Select numerical method(s). Examples include:

- Method of lines.
- Finite differencing.
- Pseudo-spectral methods.
- FMR, AMR.
- Summation by parts.
- Multipatch.
- High-Resolution-Shock-Capturing.
- Discontinuous Galerkin.



Szilágyi et.al 2009.

Then use whatever methods of numerical analysis are available to see if approximation method can succeed. Often no guarantee.

A recipe for Numerical Relativity 5

5. Implemention of numerical method. Some considerations:

- Bad design decisions can hurt if you have to rewrite.
- Language, libraries, OSS. Reinventing bad if avoidable.
- Find code monkey, or better become one.
- Test *everything*. Document. Use weak field solutions.

Andreas Weyhausen. Used with permission.

- Optimization. MPI, Open MP, GPU computing. Optimize. Make code fast

All of this is hard and the skill is *very* important both in NR and elsewhere.

A recipe for Numerical Relativity 6

6. Evaluate errors in the numerical work. Perform convergence tests. Solve problem at higher resolution, comparing difference between data. If errors improve with resolution, but not sufficient:

- Go back and run at a higher resolution. But may have to beg for more supercomputer time.
- Develop more accurate methods.
- Make do.

Numerical Relativists spend their lives doing this; *importance can not be overstated.* Without trustworthy error estimates data worthless. Numerical analysis vs. computational physics.

A recipe for Numerical Relativity 7

7. Physical interpretation of the results. NR meets mathematical relativity, geometry or astrophysics. Need to manipulate data; visualization, postprocessing. Examples:

- GW extraction.
- Event horizons.
- Conformal diagrams.
- Pretty pictures to impress students.

Marcus Thierfelder.

General comments:

- Real world means compromise.
- NR multi-disciplinary.

A case study: critical phenomena in gravitational collapse

In the early 1990s [Matt Choptuik, PRL 70, 9-12 (1993)]. NR was fundamental in the discovery of previously unknown phenomena in gravitation.

1. Physical problem: black-hole formation from collapsing matter.
2. Formulation. Spherical symmetry. Constrained evolution.
3. PDEs analysis: Wave equation coupled to some ODEs.
4. Numerics: adaptive mesh-refinement.
5. Implementation.
6. Evaluate errors.
7. At threshold of BH formation, oscillations occur on ever smaller scales with same structure. Generates singularity *not* contained in an event horizon. Naked singularity!

What has numerical relativity taught us?

Let's look at the 'most important' problem:

1. Physical problem. Compact binaries. (Black holes here).
2. Formulation. (i). moving-puncture (ii). GHG.
3. PDEs analysis: (i). Good IVP. (ii). Good IBVP.
4. Numerics: (i). Moving box MR. (ii). Pseudospectral.
5. Implementation. (i). ET, BAM, GRChombo.
   (ii). SpEC(TRE), bamps.
6. Evaluate errors. NRAR. 2013. SpEC: phase error $\Delta\phi \lesssim 0.5$.
   15 orbits.
7. **Results?**

What has numerical relativity taught us?

7. **Results?** Quantitative!

- 2005. *Vanilla setup:* $\sim 3\%$ of mass emitted as GWs.
- 2007-2010. Superkicks! Recoils $\sim 10^3$km/s.
- 2008. Penrose limit ok.
- PN unreasonably good.
- Accurate models (EOB, Phenom).
- **Detection!**

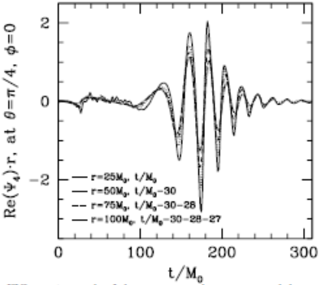


FIG. 3: A sample of the gravitational waves emitted during the merger, as estimated by the Newman-Penrose scalar Ψ_4

Frans Pretorius 2005.

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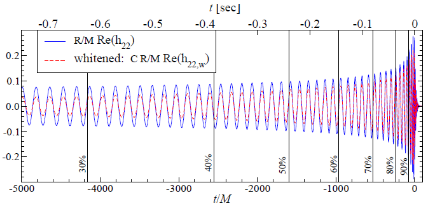

Hinder et. al. 2013.

Many open problems throughout 1-7! ☺