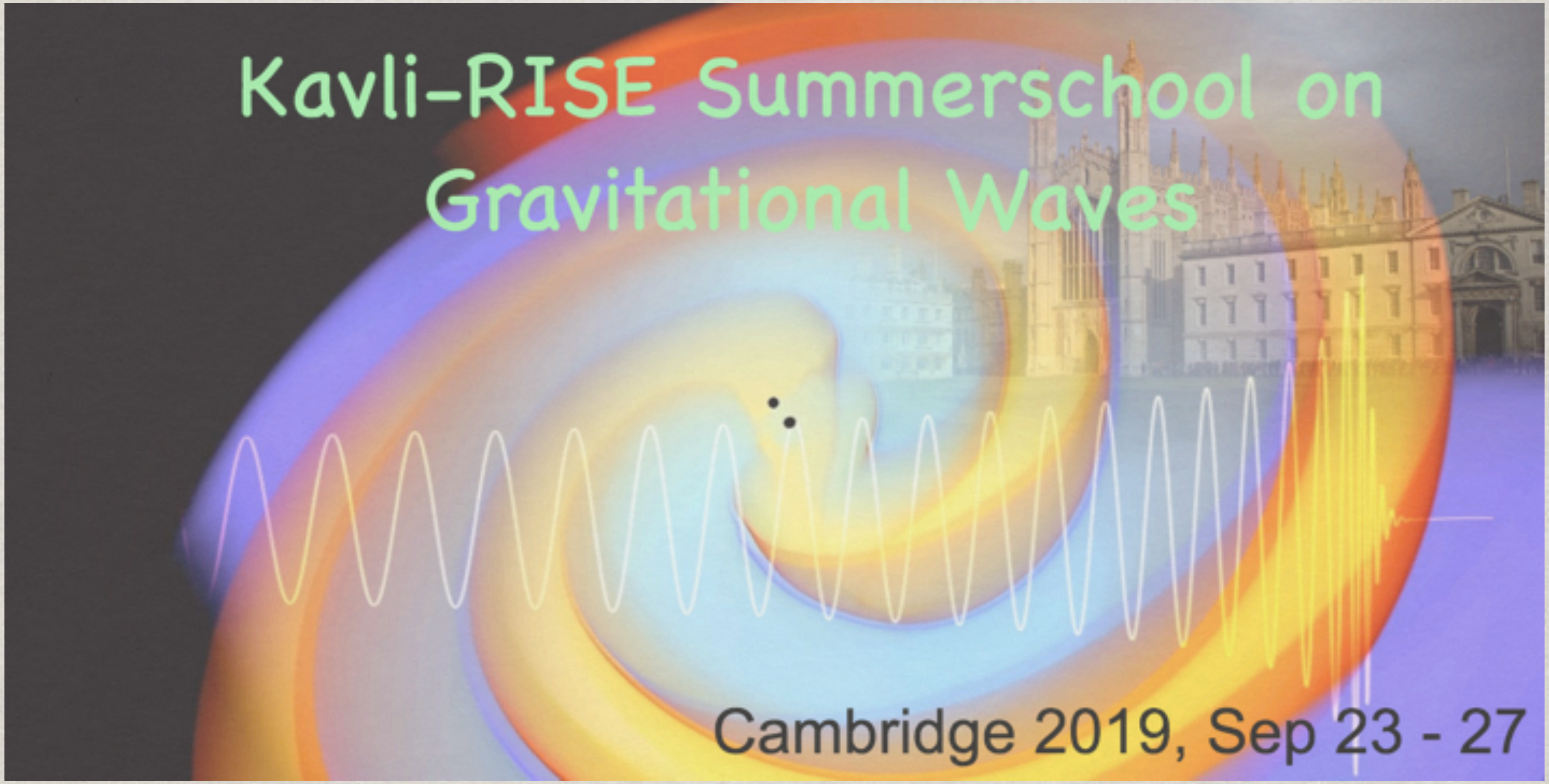
Black holes and fundamental fields II



C. Herdeiro

U. Aveiro and CENTRA
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Lecture plan:

a) Introduction: the simplicity of black holes

b) Story I: Linear analysis and new dof (“hair”)

c) Story II: Non-linear analysis - new black holes and solitons

d) Discussion

1963: **Kerr's solution**

Phys. Rev. Lett. 11 (1963) 237-238

GRAVITATIONAL FIELD OF A SPINNING MASS AS AN EXAMPLE OF ALGEBRAICALLY SPECIAL METRICS

Roy P. Kerr*

University of Texas, Austin, Texas and Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio
(Received 26 July 1963)

Goldberg and Sachs[1] have proved that the algebraically special solutions of Einstein's empty-space field equations are characterized by the existence of a geodesic and shear-free ray congruence, k_μ. Among these spaces are the plane-fronted waves and the Robinson-Trautman metrics[2] for which the congruence has nonvanishing divergence, but is hypersurface orthogonal.

where ζ is a complex coordinate, a dot denotes differentiation with respect to u, and the operator D is defined by

$$D = \partial/\partial\zeta - \Omega\partial/\partial u.$$

P is real, whereas Ω and m (which is defined to be $m_1 + im_2$) are complex. They are all independent of the coordinate r. Δ is defined by

$$ds^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma}dt^2 - 2a\sin^2\theta\frac{(r^2+a^2-\Delta)}{\Sigma}dtd\phi + \left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

$$\Sigma = r^2 + a^2\cos^2\theta$$
$$\Delta = r^2 - 2GMr + a^2$$

(in the coordinates introduced by Robert H. Boyer and Richard W. Lindquist, in 1967, J. Math. Phys. 8 (1967) 265)

1967: **Israel's theorem**

PHYSICAL REVIEW VOLUME 164, NUMBER 5 25 DECEMBER 1967

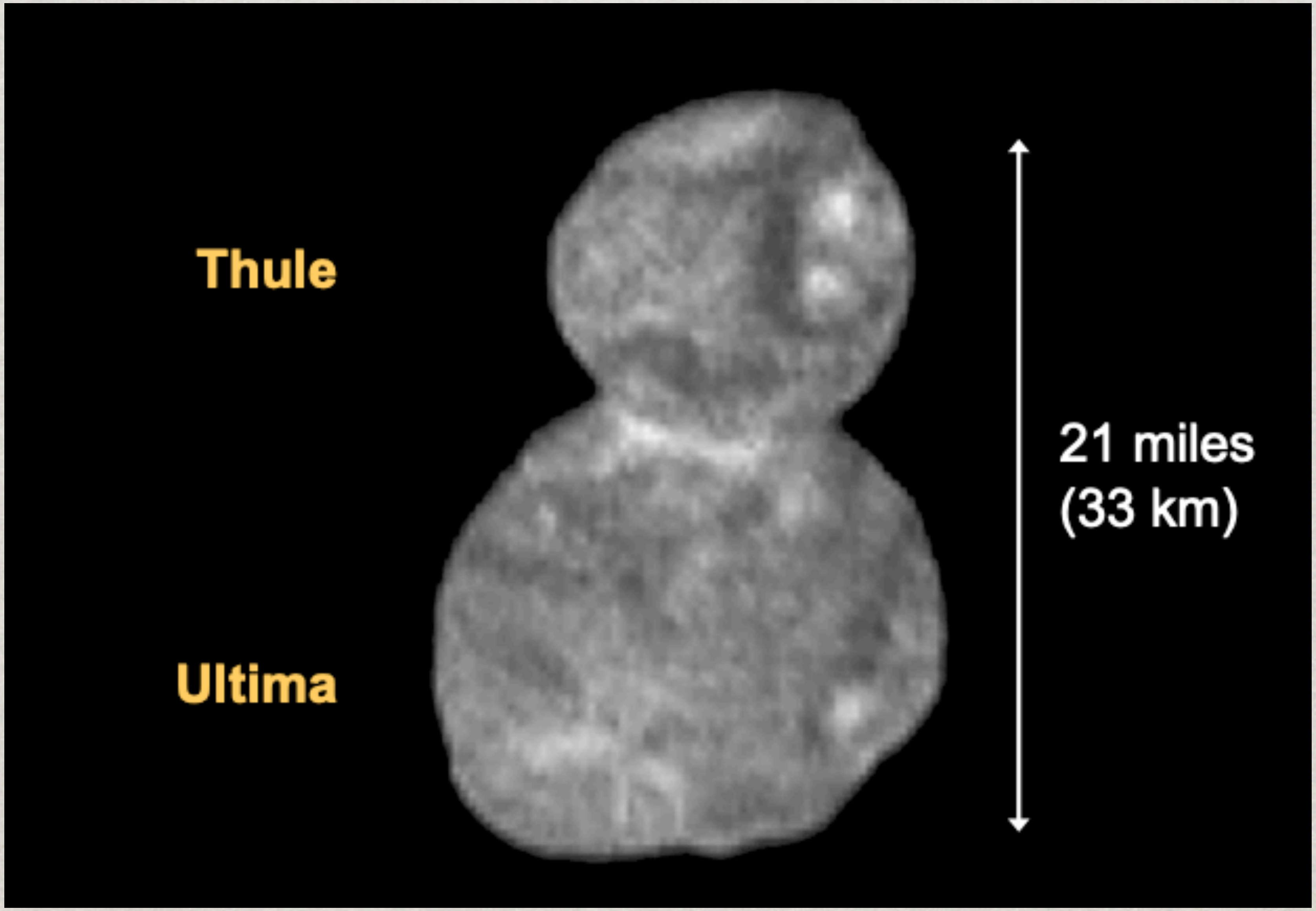
Event Horizons in Static Vacuum Space-Times

WERNER ISRAEL
Mathematics Department, University of Alberta, Alberta, Canada
and
Dublin Institute for Advanced Studies, Dublin, Ireland
(Received 27 April 1967)

The following theorem is established. Among all static, asymptotically flat vacuum space-times with closed simply connected equipotential surfaces $g_{00}=\text{constant}$, the Schwarzschild solution is the only one which has a nonsingular infinite-red-shift surface $g_{00}=0$. Thus there exists no static asymmetric perturbation of the Schwarzschild manifold due to internal sources (e.g., a quadrupole moment) which will preserve a regular event horizon. Possible implications of this result for asymmetric gravitational collapse are briefly discussed.

Israel's theorem:
An asymptotically flat static vacuum spacetime that is non-singular on and outside an event horizon, must be isometric to the Schwarzschild spacetime.

The snowman asteroid



1967-...: **The electro-vacuum uniqueness theorems**

Axisymmetric Black Hole Has Only Two Degrees of Freedom

B. Carter
Institute of Theoretical Astronomy, University of Cambridge, Cambridge CB3 0EZ, England
(Received 18 December 1970)

A theorem is described which establishes the claim that in a certain canonical sense the Kerr metrics represent "the" (rather than merely "some possible") exterior fields of black holes with the corresponding mass and angular-momentum values.

Phys. Rev. Lett. 26 (1971) 331-333

Vacuum:

$$\mathcal{S} = \frac{1}{16\pi}\int d^4x\sqrt{-g}R$$

Kerr Kerr 1963

Uniqueness Israel 1967; Carter 1971; D.C. Robinson, Phys. Rev. Lett. 34, 905 (1975).

1971: Wheeler and Ruffini coin the expression "a black hole has no hair"

The collapse leads to a black hole endowed with mass and charge and angular momentum but, so far as we can now judge, no other adjustable parameters: "A black hole has no hair." Make one black hole out of matter; another, of the same mass, angular momentum, and charge, out of antimatter. No one has ever been able to propose a workable way to tell which is which. Nor is any way known to distinguish either from a third black hole, formed by collapse of a much smaller amount of matter and then built up to the specified mass and angular momentum by firing in enough photons, or neutrinos, or gravitons. And on an equal footing is a fourth black hole, developed by collapse of a cloud of radiation altogether free from any "matter."

Electric charge is a distinguishable quantity because it carries a long-range force (conservation of flux; Gauss's law). Baryon number and strangeness carry no such long-range force. They have no Gauss's law. It is true that no attempt to observe a change in baryon number has ever succeeded. Nor has anyone ever been able to give a convincing reason to expect a direct and spontaneous violation of the principle of conservation of baryon number. In gravitational collapse, however, that principle is not directly violated; it is transcended. It is transcended because in collapse one loses the possibility of measuring baryon number, and therefore this quantity can not be well defined for a collapsed object. Similarly, strangeness is no longer conserved.

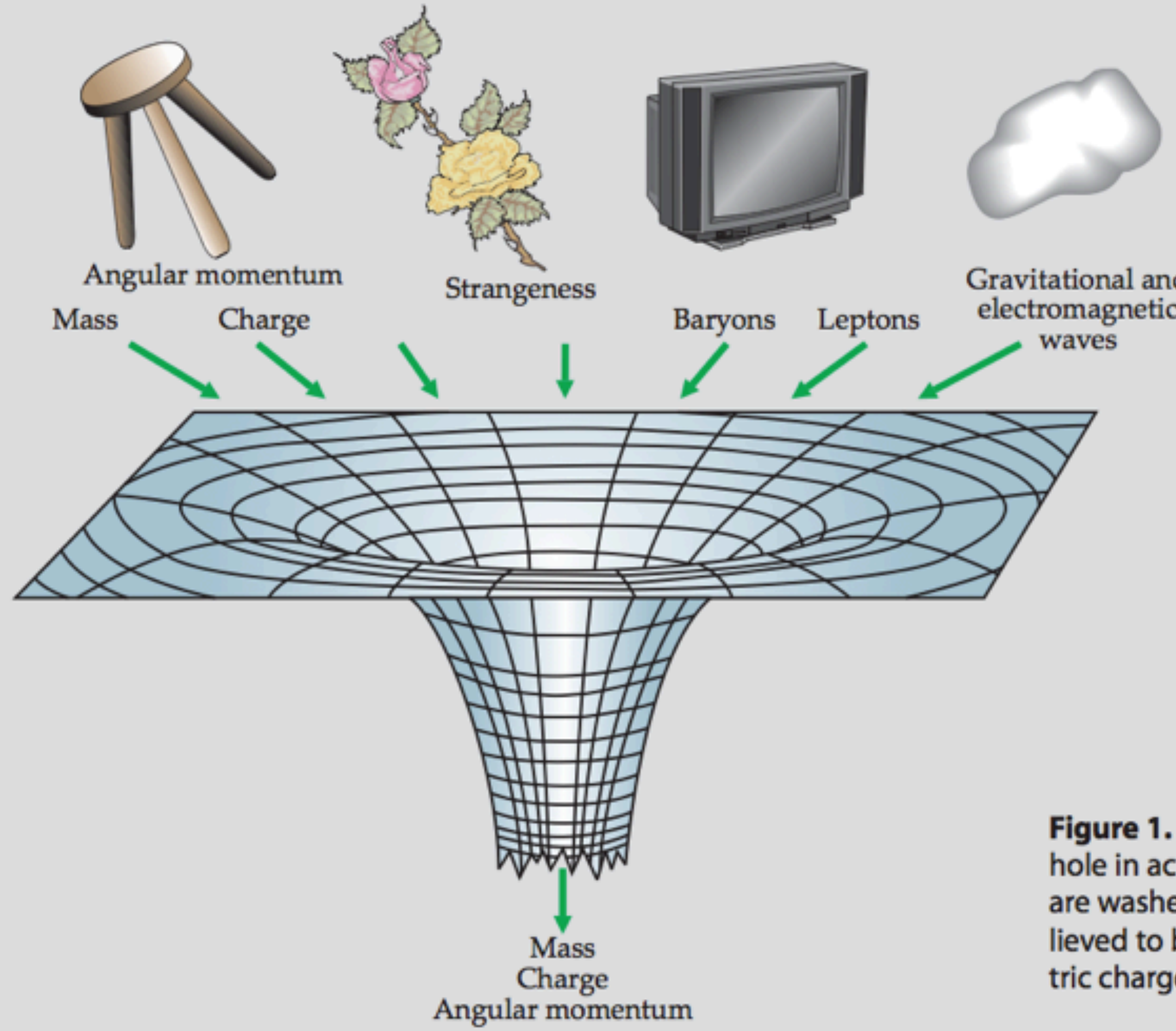


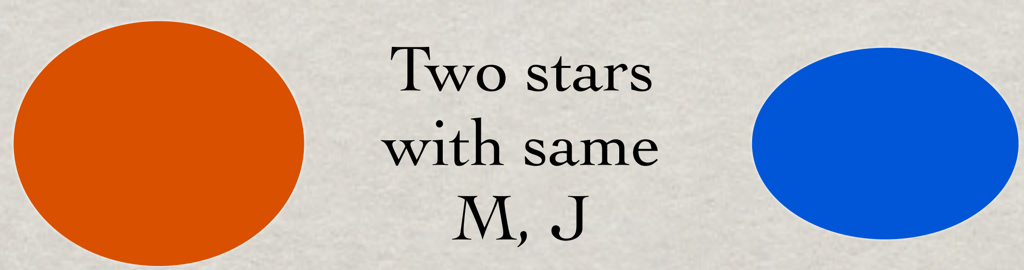
Figure 1. F
hole in acti
are washed
lieved to b
tric charge,

R. Ruffini and John Wheeler, "Introducing the black hole", Physics Today, January 1971, Pages 30-41

The "no-hair" original idea (1971):

collapse leads to equilibrium black holes uniquely determined by M,J,Q - asymptotically measured quantities subject to a Gauss law and no other independent characteristics (hair)

The idea is motivated by the uniqueness theorems and indicates black holes are very special objects



Can have a different mass quadrupole, etc...

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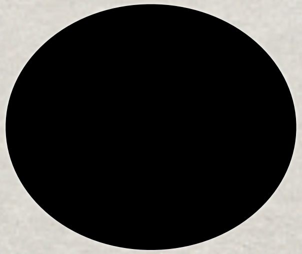
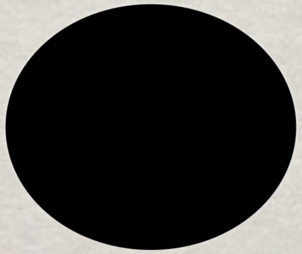


... but two black holes with same M, J...



...must be exactly equal...

Elegant multipoles formula (for the Kerr solution):

R. O. Hansen, J. Math. Phys. 15 (1974) 46

$$M_\ell + iS_\ell = M(ia)^\ell$$

Lesson...

Black holes in electrovacuum GR
may have multipoles,
but have no "multipolar freedom"

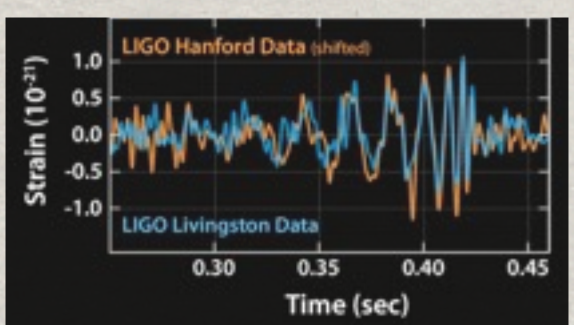
The **Kerr hypothesis** states that astrophysical black holes, when near equilibrium, are well described by the Kerr metric.

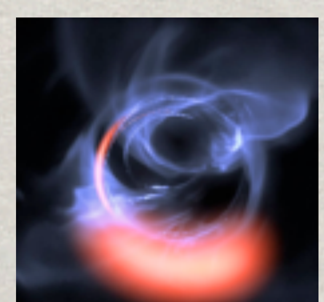
This is a very economical scenario:
the very same "object" spans (at least) 10 orders of magnitude!

Stellar mass range

Supermassive range

0 2 4 6 8 10 $\log\left(\frac{M}{M_\odot}\right)$





Let us theoretically test the
Kerr hypothesis
adding fundamental fields
to (electro)vacuum

An intriguing possibility is
that astrophysical black holes are non-Kerr,
but only in some particular scales.

Also Thomas' talk

Lecture plan:

a) Introduction: the simplicity of black holes

b) Story I: Linear analysis and new dof (“hair”)

c) Story II: Non-linear analysis - new black holes and solitons

d) Discussion

From Vitor's lecture:

Static:

$$\frac{\partial \Phi}{\partial t} = 0$$

A solution to (A) is the Schwarzschild solution,

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

To solve (B) decompose

$$\psi = \sum_{lm} \frac{\Phi(t,r)}{r} Y_{lm}(\theta,\phi)$$

$$\frac{\partial^2}{\partial r_*^2}\Phi - \cancel{\frac{\partial^2}{\partial t^2}\Phi} - V\Phi = 0, \text{ with}$$

$$\frac{\partial r_*}{\partial r} = \frac{1}{f} = \frac{1}{1-\frac{2M}{r}} \quad \& \quad V = f\left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3} + \mu^2\right]$$

• Static solutions I

$$\left[\left(1-\frac{2M}{r}\right)\phi'\right]' - V\phi = 0$$

Multiply by ϕ^* and integrate from $2M$ to ∞:

$$\int_{2M}^{\infty} \left[\left(1-\frac{2M}{r}\right)\phi'\right]'\phi^* - V|\phi|^2 = 0$$

$$\left[\left(1-\frac{2M}{r}\right)\phi'\phi^*\right]_{2M}^{\infty} - \int_{2M}^{\infty}\left[\left(1-\frac{2M}{r}\right)|\phi'|^2 + V|\phi|^2\right] = 0$$

$$\Rightarrow \Phi = 0$$

(linear) no-scalar-hair theorem

However, for a GR solution $(g_{\mu\nu}, \Phi)$,
"static" does not necessarily require:

$$\frac{\partial \Phi}{\partial t} = 0$$

It requires:

$$\mathcal{L}_k g_{\mu\nu} = 0 \;\Rightarrow\; \mathcal{L}_k T_{\mu\nu} = 0 \;\cancel{\Leftrightarrow}\; \mathcal{L}_k \Phi = 0$$

For a complex scalar field (say, massive):

$$T_{\alpha\beta} = \Phi^*_{,\alpha}\Phi_{,\beta} + \Phi^*_{,\beta}\Phi_{,\alpha} - g_{\alpha\beta}\left[\frac{1}{2}g^{\gamma\delta}(\Phi^*_{,\gamma}\Phi_{,\delta} + \Phi^*_{,\delta}\Phi_{,\gamma}) + \mu^2\Phi^*\Phi\right]$$

Staticity is compatible with a harmonic time dependence. In adapted coordinates:

$$\Phi = e^{-i\omega t}\phi(\vec{r})$$

Lesson...

A matter field does not have to be invariant under the spacetime isometries.

In particular, it does not need to be "time independent" in a static (stationary) spacetime.

Exercise!

Does the linear no scalar hair theorem hold if one admits a harmonic time dependence for the scalar field?

A complex, massive, test scalar field on Schwarzschild

2.1 A complex, massive, test scalar field on Schwarzschild

Question: Are there "bound states", in the sense of quantum mechanics, of a scalar field around a Schwarzschild black hole?

2.1 A complex, massive, test scalar field on Schwarzschild

Recall the Hydrogen atom in non-relativistic quantum mechanics (no-spin):

$$i\hbar\frac{\partial}{\partial t}\Psi(t,r,\theta,\phi) = \left[-\frac{\hbar^2}{2\mu}\Delta + V(r)\right]\Psi(t,r,\theta,\phi) \qquad V(r) = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

One looks for stationary states:

$$\Psi(t,r,\theta,\phi) = e^{-iwt}\frac{R(r)}{r}Y_\ell^m(\theta,\phi) \qquad Y_\ell^m(\theta,\phi) = P_\ell^m(\cos\theta)e^{im\phi}$$

$$\left[\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + \underbrace{\ell(\ell+1)}_{\text{Separation constant}}\sin^2\theta - m^2\right]P_\ell^m(\cos\theta) = 0$$

defines the associated Legendre polynomials (and the complete spherical harmonics)

$$\left\{-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \left[-\frac{e^2}{4\pi\epsilon_0 r} + \frac{\hbar^2\overbrace{\ell(\ell+1)}}{2\mu r^2}\right]\right\}R(r) = ER(r)$$

Effective 1D Schrödinger problem

2.1 A complex, massive, test scalar field on Schwarzschild

The radial equation can be rewritten as:

$$r^2 \frac{d^2}{dr^2} R(r) = \left[-\frac{2\mu E}{\hbar^2} r^2 - \frac{\mu e^2}{2\pi\epsilon_0 \hbar^2} r + \ell(\ell+1) \right] R(r)$$

Which has the form of the Whittaker equation (confluent hypergeometric type):

$$z^2 \frac{d^2}{dz^2} W(z) = \left[\frac{z^2}{4} - kz + \left(p^2 - \frac{1}{4} \right) \right] W(z)$$

with: $$z = \frac{\sqrt{-8\mu E}}{\hbar} r \,, \qquad k = \frac{e^2}{4\pi\epsilon_0 \hbar} \sqrt{-\frac{\mu}{2E}} \,, \qquad p = \ell + \frac{1}{2}$$

Whittaker's equation is solved in terms of a power series:

$$W(z) = z^{p+1/2} e^{-z/2} \sum_{n=0}^{\infty} b_n z^n$$

Exercise!

Insert the power series in Whittaker's equation and obtain the quantization condition.

2.1 A complex, massive, test scalar field on Schwarzschild

Radial equation can be solved by a power series (leading to a 2-term recurrence relation); boundedness of the wave function leads to the condition:

$$k = \frac{1}{2} + p + n = \ell + n + 1$$

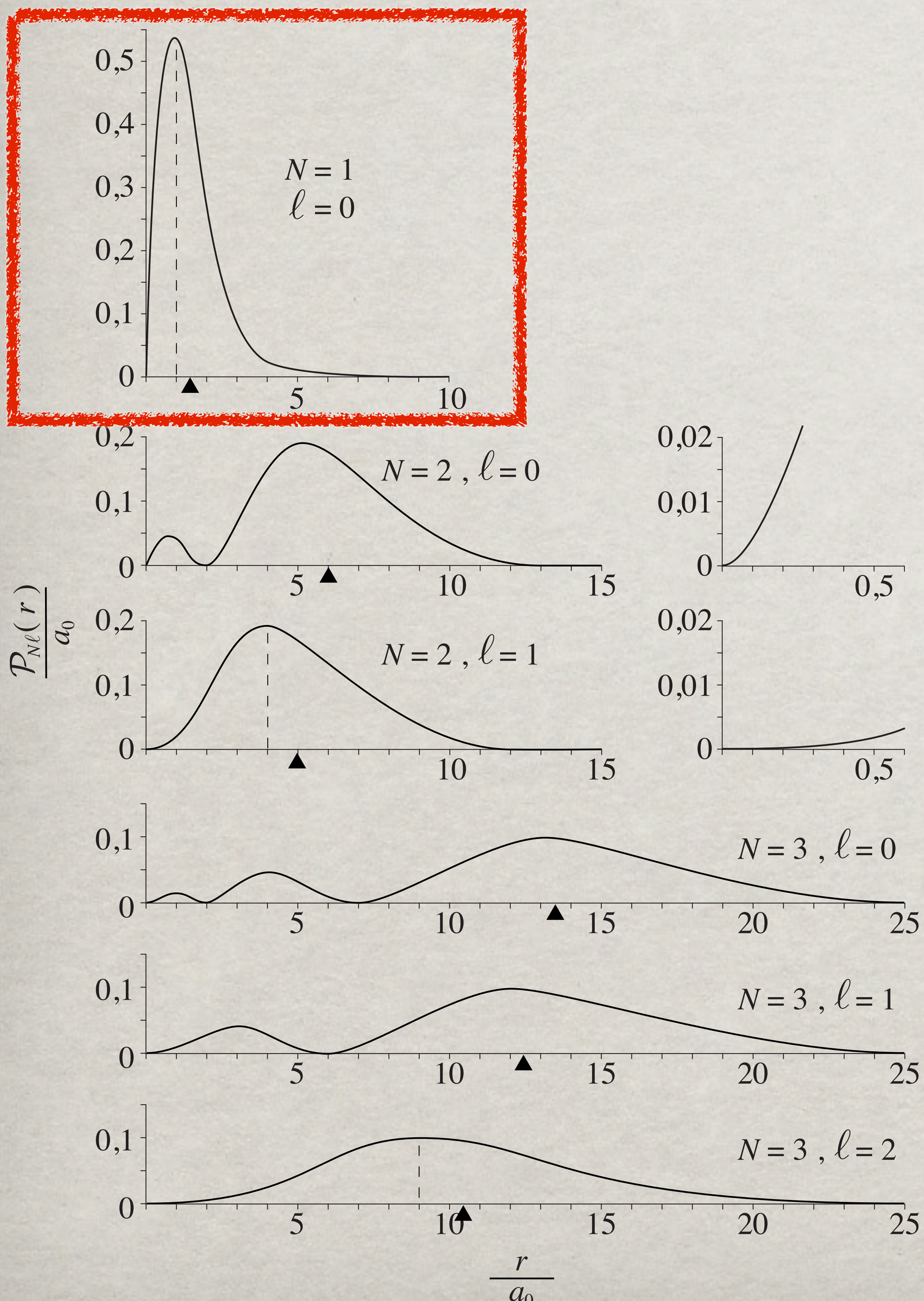
for some integer n. This quantizes the frequencies. These frequencies are **real**. The corresponding states are **bound states**.

$$w_{(\ell,n)} = -\left(\frac{e^2}{4\pi\epsilon_0\hbar}\right)^2 \frac{\mu}{2\hbar(\ell+n+1)^2}$$

Observe:

1) n= overtone number (counts nodes of the radial function);
2) spherical symmetry implies no dependence on m;
3) spectrum only depends on the principal quantum number $N \equiv n + \ell + 1$. This because there is a hidden symmetry for this problem [SO(4)].

2.1 A complex, massive, test scalar field on Schwarzschild



Radial probability density:

2.1 A complex, massive, test scalar field on Schwarzschild

Now look for **stationary bound states** of a:
- **massive**, to guarantee an exponential fall-off;
- **complex**, to have a harmonic time dependence at the level of the field, and no time dependence in the energy-momentum tensor;

Klein-Gordon scalar field on the Schwarzschild background:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{(1 - \frac{2GM}{r})} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$\Box\Phi = \mu^2\Phi$$

$$\Phi(t, r, \theta, \phi) = \sum_{\ell,m} Y_\ell^m(\theta, \phi) e^{-i\omega t} \frac{R(r)}{r},$$

In order to transform this problem into an effective 1D Schrödinger problem one needs also to consider the Regge-Wheeler radial coordinate:

$$dr^* = \frac{dr}{1 - 2M/r}$$

2.1 A complex, massive, test scalar field on Schwarzschild

One obtains the effective 1D Schrödinger problem:

$$\left[-\frac{d^2}{d(r^*)^2}+V_{eff}(r)\right]R(r)=\omega^2 R(r) \qquad V_{eff}=\left(1-\frac{2M}{r}\right)\left(\frac{2M}{r^3}+\frac{\ell(\ell+1)}{r^2}+\mu^2\right)$$

M is a scale;
parameters are: ℓ, μ

A potential well is possible. But there is a **crucial difference** with respect to the standard bound states problems in Quantum Mechanics: the boundary condition at the horizon. Near the horizon:

$$V_{eff}\simeq 0 \quad \Rightarrow \quad -\frac{d^2R(r)}{d(r^*)^2}\simeq\omega^2 R(r) \quad \Rightarrow \quad R(r)\simeq e^{\pm iwr^*}$$

2.1 A complex, massive, test scalar field on Schwarzschild

At the horizon we impose only ingoing modes (minus sign). Physically, thus, we expect no real bound states to exist, since there is an energy flux into the black hole. The near horizon solution can be rewritten:

$$R(r) \simeq e^{-iwr^*} \simeq \left(\frac{r-2M}{2M}\right)^{-2M\omega i}$$

At infinity, to leading order (zeroth order in M):

$$\frac{d^2R(r)}{dr^2} \simeq (\mu^2-\omega^2)R(r) \quad \Rightarrow \quad R(r) \simeq e^{\pm\sqrt{\mu^2-\omega^2}\,r}$$

choose decaying solution (minus sign) for a gravitationally bound state. Observe the bound state condition $w<\mu$

To the next order in M/r:

$$R(r) = \frac{e^{-\sqrt{\mu^2-\omega^2}\,r}}{r^{\frac{\mu^2-2\omega^2}{\sqrt{\mu^2-\omega^2}}M}}$$

Exercise!

Obtain this.

2.1 A complex, massive, test scalar field on Schwarzschild

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$$R(r) \simeq e^{-iwr^*} \simeq \left(\frac{r-2M}{2M}\right)^{-2M\omega i} \qquad\qquad r \to 2M$$

To the next order in M/r:

$$R(r) = \frac{e^{-\sqrt{\mu^2-\omega^2}\,r}}{r^{\frac{\mu^2-2\omega^2}{\sqrt{\mu^2-\omega^2}}M}} \qquad\qquad r \to \infty$$

The radial equation can be tackled, for instance, using Leaver's method E. W. Leaver, Proc. Roy. Soc. Lond. A 402 (1985) 285:

$$R(r) = (r-2M)^{-2M\omega i}\, r^{2M\omega i+\chi}\, e^{-\sqrt{\mu^2-\omega^2}\,r} \sum_{n=0}^{\infty} a_n \left(\frac{r-2M}{r}\right)^n \qquad\qquad \chi = -\frac{\mu^2-2\omega^2}{\sqrt{\mu^2-\omega^2}}M$$

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This leads to a 3-term recurrence relation:

Exercise!
Obtain the coefficients.

$$\alpha_0 a_1 + \beta_0 a_0 = 0 \qquad \Rightarrow \quad \frac{\beta_0}{\alpha_0} = -\frac{a_1}{a_0} = \frac{\gamma_1}{\beta_1 + \alpha_1 \dfrac{a_2}{a_1}} = \cdots = \frac{\gamma_1}{\beta_1 - \dfrac{\alpha_1\gamma_2}{\beta_2 - \dfrac{\alpha_2\gamma_3}{\beta_3 - \ldots}}}$$ [redacted]

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0$$

Continued fraction

2.1 A complex, massive, test scalar field on Schwarzschild

Thus the frequencies are determined by solving (to the desired accuracy):

$$F(\omega) \equiv \frac{\beta_0}{\alpha_0} - \cfrac{\gamma_1}{\beta_1 - \cfrac{\alpha_1\gamma_2}{\beta_2 - \cfrac{\alpha_2\gamma_3}{\beta_3 - \ldots}}} = 0$$

Thus:

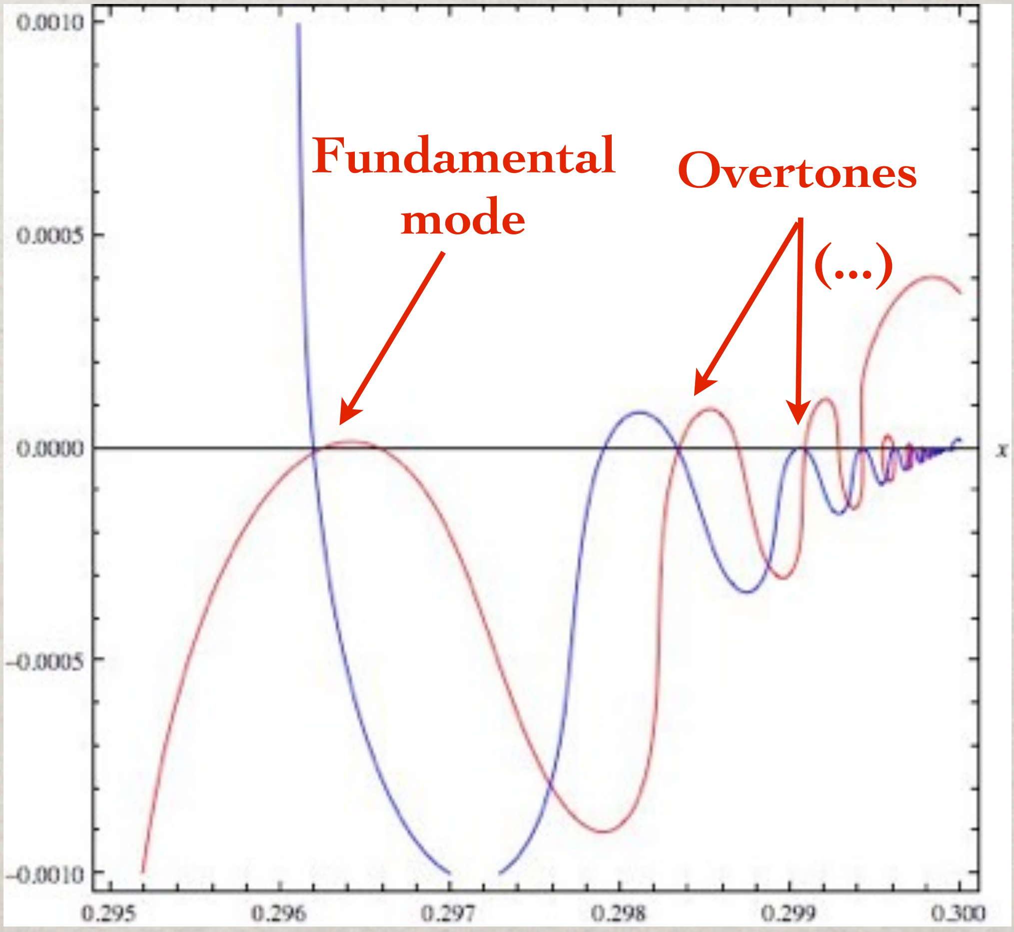
$$\mathcal{R}[F(w)] = 0 \qquad \mathcal{I}[F(w)] = 0$$

Each of them is a surface:

$$z_1 = z_1(x,y) = 0 \qquad z_2 = z_2(x,y) = 0$$

$$w = x + iy$$

This leads to two curves on the complex w plane; intersection points are solutions:



2.1 A complex, massive, test scalar field on Schwarzschild

Some results for the fundamental mode (frequencies and masses in units of M):

$$\omega M \to \omega \ , \quad \mu M \to \mu$$

$\ell = 1$

μ	ω
0.1	$0.09987 - 1.5182 \times 10^{-11} i$
0.2	$0.19895 - 4.0586 \times 10^{-8} i$
0.3	$0.29619 - 9.4556 \times 10^{-6} i$
0.4	$0.38955 - 5.6274 \times 10^{-4} i$
0.5	$0.47759 - 5.5441 \times 10^{-3} i$

$\ell = 2$

μ	ω
0.1	$0.09994 - 8.6220 \times 10^{-17} i$
0.2	$0.19954 - 5.9249 \times 10^{-14} i$
0.3	$0.29844 - 4.9002 \times 10^{-11} i$
0.4	$0.39619 - 1.1703 \times 10^{-8} i$
0.5	$0.49219 - 1.2271 \times 10^{-6} i$
0.6	$0.58541 - 6.9974 \times 10^{-5} i$
0.7	$0.67385 - 1.4987 \times 10^{-3} i$
0.8	$0.75788 - 8.1511 \times 10^{-3} i$

These frequencies are **complex** (observe the imaginary part is always negative). The corresponding states are ***quasi*-bound states**.

2.1 A complex, massive, test scalar field on Schwarzschild

The mass term allows gravitational trapping; but the horizon boundary condition only permits the existence of **quasi-bound states** around the Schwarzschild solution. These can be very long lived, especially for small masses. The **lifetime** is:

$$\tau \sim \frac{1}{Im(\omega)}$$

2.1 A complex, massive, test scalar field on Schwarzschild

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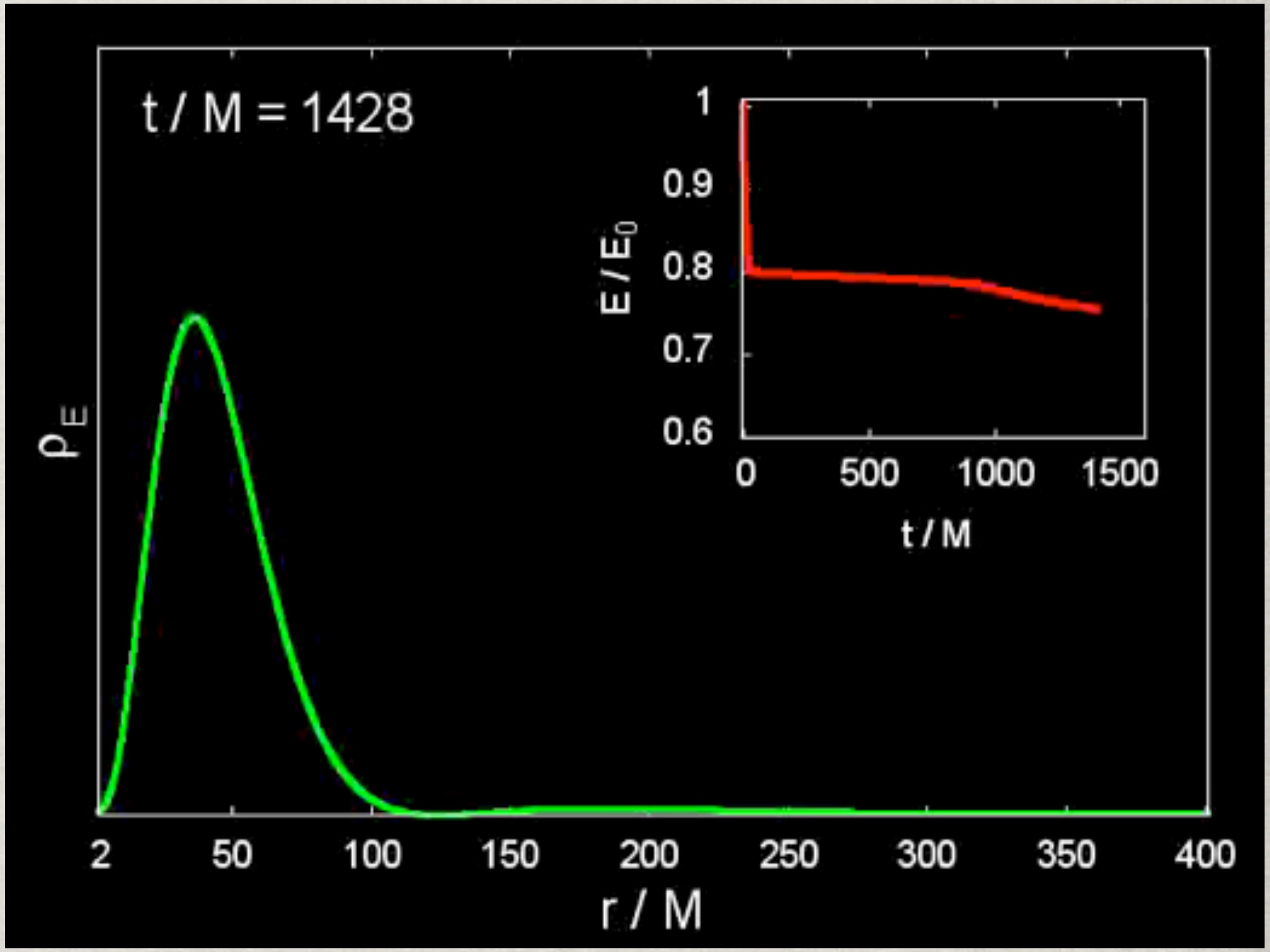


Courtesy of J. C. Degollado

These states have been called scalar "wigs": J. Barranco, A. Bernal, J. C. Degollado, A. Diez-Tejedor, M. Megevand, M. Alcubierre, D. Nunez and O. Sarbach, Phys. Rev. Lett. 109 (2012) 081102 [arXiv:1207.2153 [gr-qc]].

There is a no-scalar hair theorem for spherical static BHs with a scalar field with harmonic time dependence: Pena and D. Sudarsky, Class. Quant. Grav. 14 (1997) 3131

2.2 A complex, massive, test scalar field on Kerr

2.2 A complex, massive, test scalar field on Kerr

A similar computation can be done to obtain quasi-bound states of a massive, complex Klein-Gordon scalar field on the Kerr background:

$$ds^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma}dt^2 - 2a\sin^2\theta\frac{(r^2+a^2-\Delta)}{\Sigma}dtd\phi$$
$$+\left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

$$\Sigma = r^2 + a^2\cos^2\theta$$
$$\Delta = r^2 - 2Mr + a^2$$

$$\Box\Phi = \mu^2\Phi$$

$$\Phi(t,r,\theta,\phi) = \sum_{\ell,m} e^{im\phi}S_{\ell m}(\theta)e^{-i\omega t}R_{\ell m}(r).$$

2.2 A complex, massive, test scalar field on Kerr

One can separate variables and obtain two linear ODEs.

1) The first one defines the **spheroidal harmonics**:

M. Abramowitz, and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Dover, New York, 1965; E. Berti, V. Cardoso, and M. Casals, Phys. Rev. D 73 (2006), 024013.

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS_{\ell m}}{d\theta}\right)+\left[a^2(\omega^2-\mu^2)\cos^2\theta-\frac{m^2}{\sin^2\theta}+\Lambda_{\ell m}\right]S_{\ell m}=0.$$

new term, compared with associated Legendre equation(ALE)

Separation constant, which for the ALE was $\ell(\ell+1)$

2.2 A complex, massive, test scalar field on Kerr

2) The second one defines the **radial part:**

$$\frac{d}{dr}\left(\Delta\frac{dR_{\ell m}}{dr}\right)+\left[\frac{\omega^2(r^2+a^2)^2-4Mam\omega r+m^2a^2}{\Delta}-\left(\omega^2a^2+\mu^2r^2+\Lambda_{\ell m}\right)\right]=0$$

An analogous equation first arose in the study of the electronic spectrum of the hydrogen molecule W. G. Baber and H. R. Hassé, Proc. Camb. Phil. Soc. 25 (1935), 564; G. Jaffé, Z. Phys. A87 (1934) 535

This equation can be transformed into a singly-confluent Heun equation.

The quasi-bound state frequencies can, again, be obtained by Leaver's method S. R. Dolan, Phys. Rev. D 76 (2007) 084001 [arXiv:0705.2880 [gr-qc]]:

2.2 A complex, massive, test scalar field on Kerr

Some results for the fundamental mode (frequencies and masses in units of M):

$\mu = 0.3; \ell = 1$

a	$m=-1$	$m=0$	$m=1$
0.1	$0.29618 - 1.19213 \times 10^{-5} i$	$0.29619 - 9.39767 \times 10^{-6} i$	$0.29620 - 7.30823 \times 10^{-6} i$
0.5	$0.29613 - 2.51902 \times 10^{-5} i$	$0.29612 - 8.00351 \times 10^{-6} i$	$0.29625 - 1.66155 \times 10^{-6} i$
0.9	$0.29607 - 4.44672 \times 10^{-5} i$	$0.29620 - 4.68608 \times 10^{-6} i$	$0.29630 + 1.46971 \times 10^{-8} i$
0.95	$0.29600 - 4.70610 \times 10^{-5} i$	$0.29620 - 4.08878 \times 10^{-6} i$	$0.29630 + 2.72170 \times 10^{-8} i$

$\mu = 0.4; \ell = 1$

a	$m=-1$	$m=0$	$m=1$
0.1	$0.38948 - 6.62132 \times 10^{-4} i$	$0.38955 - 5.61203 \times 10^{-4} i$	$0.38963 - 4.67614 \times 10^{-4} i$
0.5	$0.38926 - 1.08538 \times 10^{-3} i$	$0.38955 - 5.23330 \times 10^{-4} i$	$0.39001 - 1.53007 \times 10^{-4} i$
0.9	$0.38914 - 1.52116 \times 10^{-3} i$	$0.38954 - 4.26952 \times 10^{-4} i$	$0.39045 - 4.34117 \times 10^{-6} i$
0.95	$0.38913 - 1.57507 \times 10^{-3} i$	$0.38954 - 4.09975 \times 10^{-4} i$	$0.39050 - 5.71763 \times 10^{-7} i$

Main new feature: the imaginary part can become positive. This means the mode grows, instead of decaying. There is a **superradiant instability**.

Lesson…

There is an instability of the Kerr black hole in the presence of a massive scalar field.

The same is true for a massive vector (Proca) field.

2.3 Superradiance

2.3 Superradiance

Superradiance is a radiation enhancement process. It is by no means exclusive to black hole physics, but it can occur in the scattering of bosonic fields by rotating (and also charged) black holes. R. Brito, V. Cardoso and P. Pani, "Superradiance", Lect. Notes Phys. 906 (2015) pp.1, [arXiv:1501.06570 [gr-qc]]

In black hole physics, superradiant amplification, leading to energy and angular momentum (or charge) extraction from the black hole, was first discussed:

- from a thermodynamic viewpoint J. Bekenstein, Phys. Rev. D7 (1973) 949-953;
- in the scattering of scalar J. M. Bardeen, W. H. Press and S. A. Teukolsky, Astrophys. J. 178 (1972) 347; A. Starobinski, Zh. Eksp. Teor. Fiz. 64 (1973) 48. (Sov. Phys. - JETP, 37, 28, 1973), electromagnetic and gravitational waves by a rotating black hole A. Starobinski and S. M. Churilov, Zh. Eksp. Teor. Fiz. 65 (1973) 3. (Sov. Phys. - JETP, 38, 1, 1973)

2.3 Superradiance

When the superradiantly amplified waves are confined in the vicinity of the black hole, there are multiple scatterings, leading to an exponential growth of the field amplitude. Press and Teukolsky suggested an explosive phenomenon follows, dubbed **black hole bomb** W. H. Press and S. A. Teukolsky, Nature 238 (1972) 211-212.

The model of Press and Teukolsky relied on placing a spherical "mirror" at some distance from the black hole, to confine the bosonic waves.

A natural mirror is the existence of a mass term for the bosonic field and that this leads to an instability of the Kerr solution in the presence of such fields

T. Damour, N. Deruelle, and R. Ruffini, Lett.Nuovo Cim. 15 (1976) 257- 262.

2.3 Superradiance

There is a critical frequency: $$w_c \equiv m\Omega_H + \cancel{q\Phi_H}$$

Quasi-bound states with:

- low frequencies (real part) are superradiant. They have a **positive** imaginary part and grow in time;

- high frequencies (real part) have a negative imaginary part and decay in time.

For Kerr, m=1 (in units of M):

a	ω_c
0.1	0.0250628
0.5	0.133975
0.9	0.313395
0.95	0.361974
0.97	0.390152
0.99	0.433804

Recall results for quasi-bound states (fundamental mode):

$\mu = 0.3; \ell = 1$

a	$m = -1$	$m = 0$	$m = 1$
0.1	$0.29618 - 1.19213 \times 10^{-5} i$	$0.29619 - 9.39767 \times 10^{-6} i$	$0.29620 - 7.30823 \times 10^{-6} i$
0.5	$0.29613 - 2.51902 \times 10^{-5} i$	$0.29612 - 8.00351 \times 10^{-6} i$	$0.29625 - 1.66155 \times 10^{-6} i$
0.9	$0.29607 - 4.44672 \times 10^{-5} i$	$0.29620 - 4.68608 \times 10^{-6} i$	$0.29630 + 1.46971 \times 10^{-8} i$
0.95	$0.29600 - 4.70610 \times 10^{-5} i$	$0.29620 - 4.08878 \times 10^{-6} i$	$0.29630 + 2.72170 \times 10^{-8} i$

2.3 Superradiance

Obvious (!) observation: Modes that exist **precisely** at the critical frequency have zero imaginary part and hence are **bound states** analogue to the ones in the hydrogen atom (not just quasi-bound states).

S. Hod, Phys. Rev. D86 (2012) 104026, arXiv:1211.3202 [gr-qc]

2.4 Scalar stationary clouds on Kerr(-Newman)

2.4 Scalar stationary clouds on Kerr(-Newman)

Massive Klein-Gordon field in the background of an extremal Kerr black hole. Thus:

$$ds^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma}dt^2 - 2a\sin^2\theta\frac{(r^2+a^2-\Delta)}{\Sigma}dtd\phi + \left(\frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

$$\Sigma = r^2 + a^2\cos^2\theta$$
$$\Delta = r^2 - 2Mr + a^2$$

$$\Box\Phi = \mu^2\Phi$$

$$\Phi(t,r,\theta,\phi) = \sum_{\ell,m} e^{im\phi}S_{\ell m}(\theta)e^{-i\omega t}R_{\ell m}(r).$$

with: $$a = M\ , \quad w = m\Omega_H = \frac{m}{2M}$$

The angular equation is the same as before. But the radial equation simplifies.

2.4 Scalar stationary clouds on Kerr(-Newman)

The angular equation is the same as before. But the radial equation simplifies.

With the redefinitions:

$$z = 2(r-M)\sqrt{\mu^2 - \frac{m^2}{4}} \qquad k = \frac{m^2 - 2M^2\mu^2}{\sqrt{4M^2\mu^2 - m^2}} \qquad W = \frac{r-M}{M}R_{\ell m}$$

$$p^2 = \Lambda_{\ell m} + M^2(w^2 - \mu^2) + \frac{1}{4} - 2m^2 + 2M^2\mu^2$$

Exercise!
Obtain this equation.

It becomes precisely Whittaker's equation!

$$z^2 \frac{d^2}{dz^2} W(z) = \left[\frac{z^2}{4} - kz + \left(p^2 - \frac{1}{4}\right)\right] W(z)$$

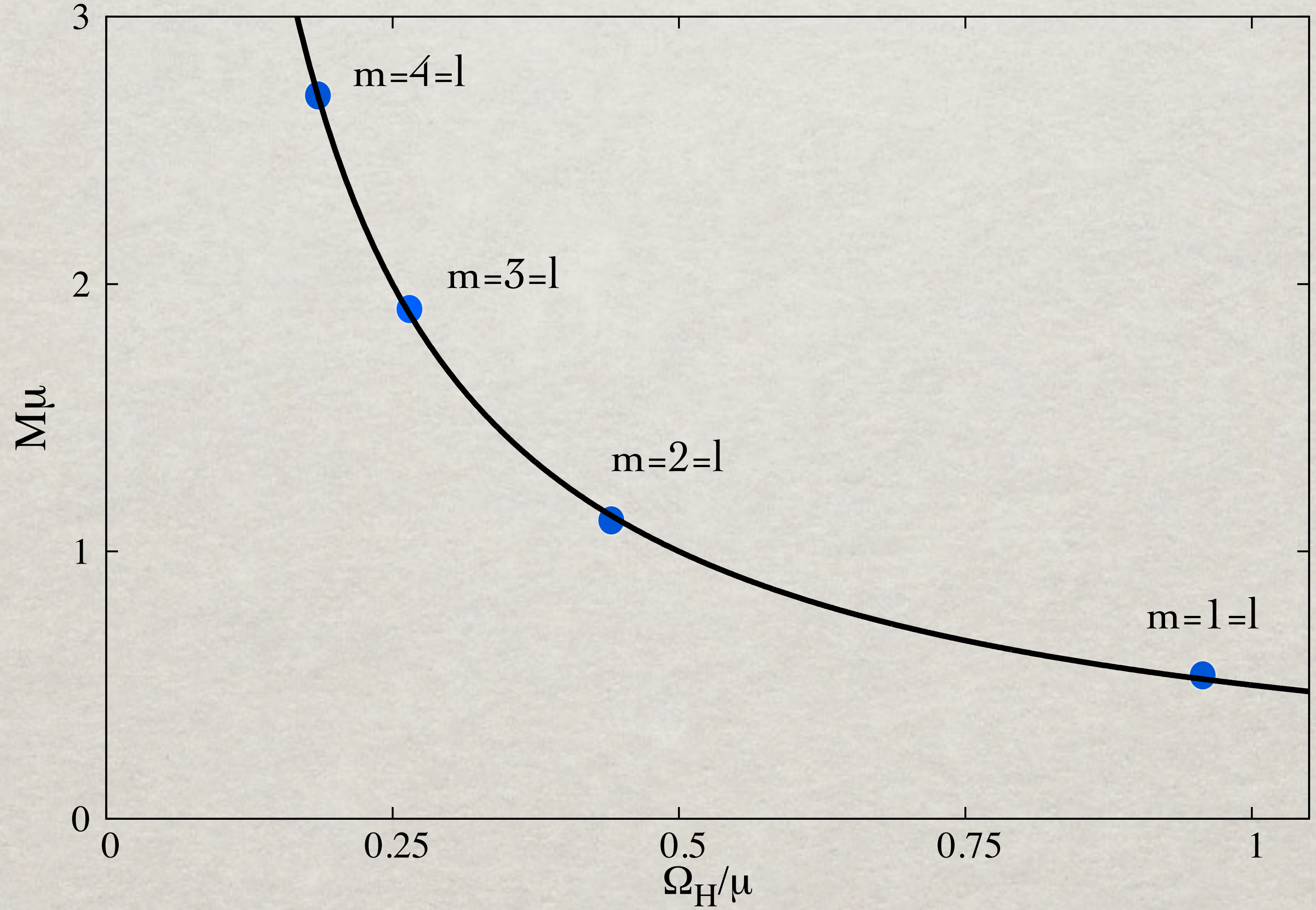
Asymptotic boundedness of the scalar field leads to the same quantization condition we saw in the Hydrogen atom:

$$k = \frac{1}{2} + p + n$$

This is now interpreted as a quantization on the black hole mass M.

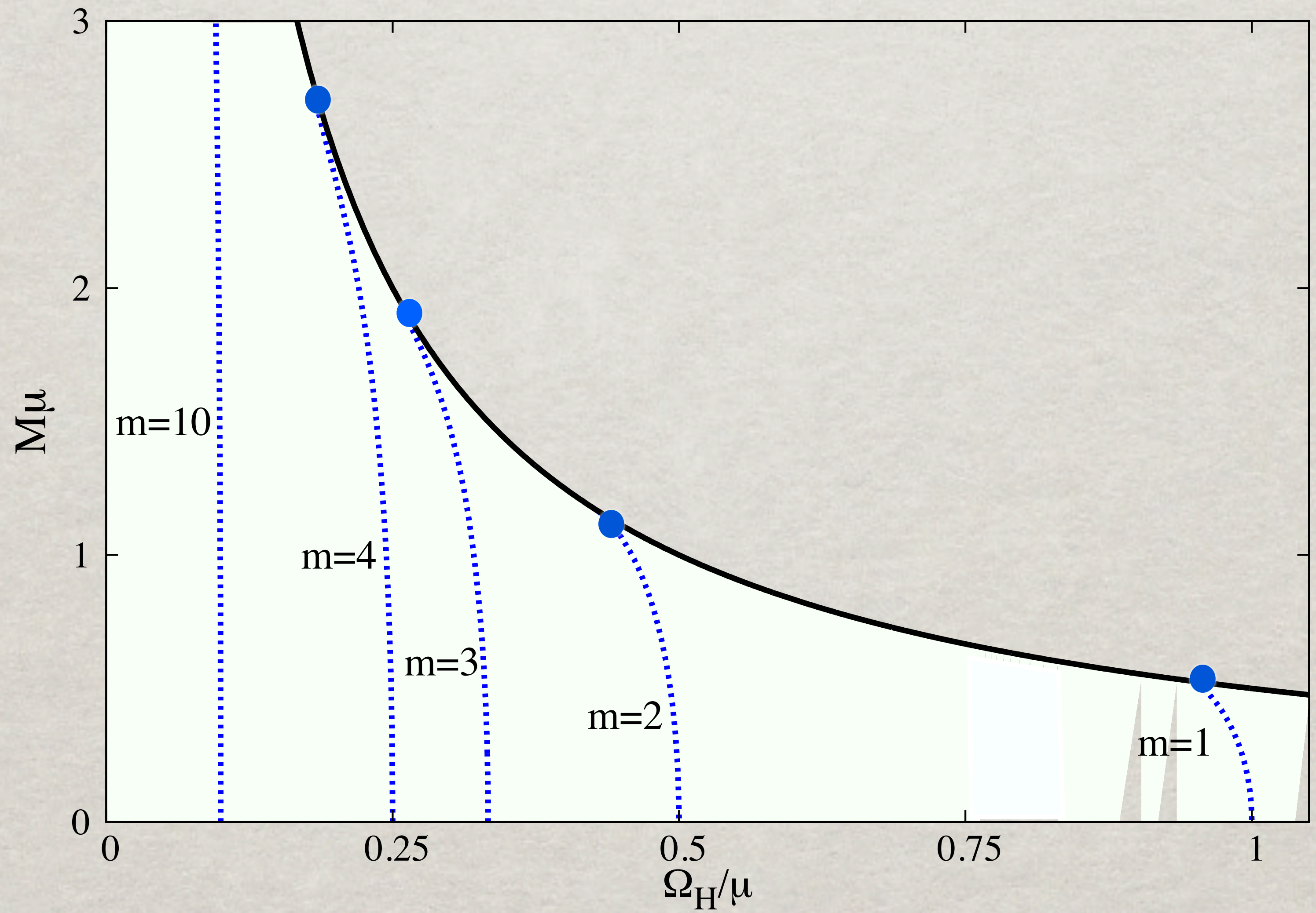
2.4 Scalar stationary clouds on Kerr(-Newman)

For each (l,m,n), there is an extremal Kerr black hole mass that admits a stationary cloud. Then, via the critical frequency condition, this defines a horizon angular velocity. S. Hod, Phys. Rev. D86 (2012) 104026, arXiv:1211.3202 [gr-qc]



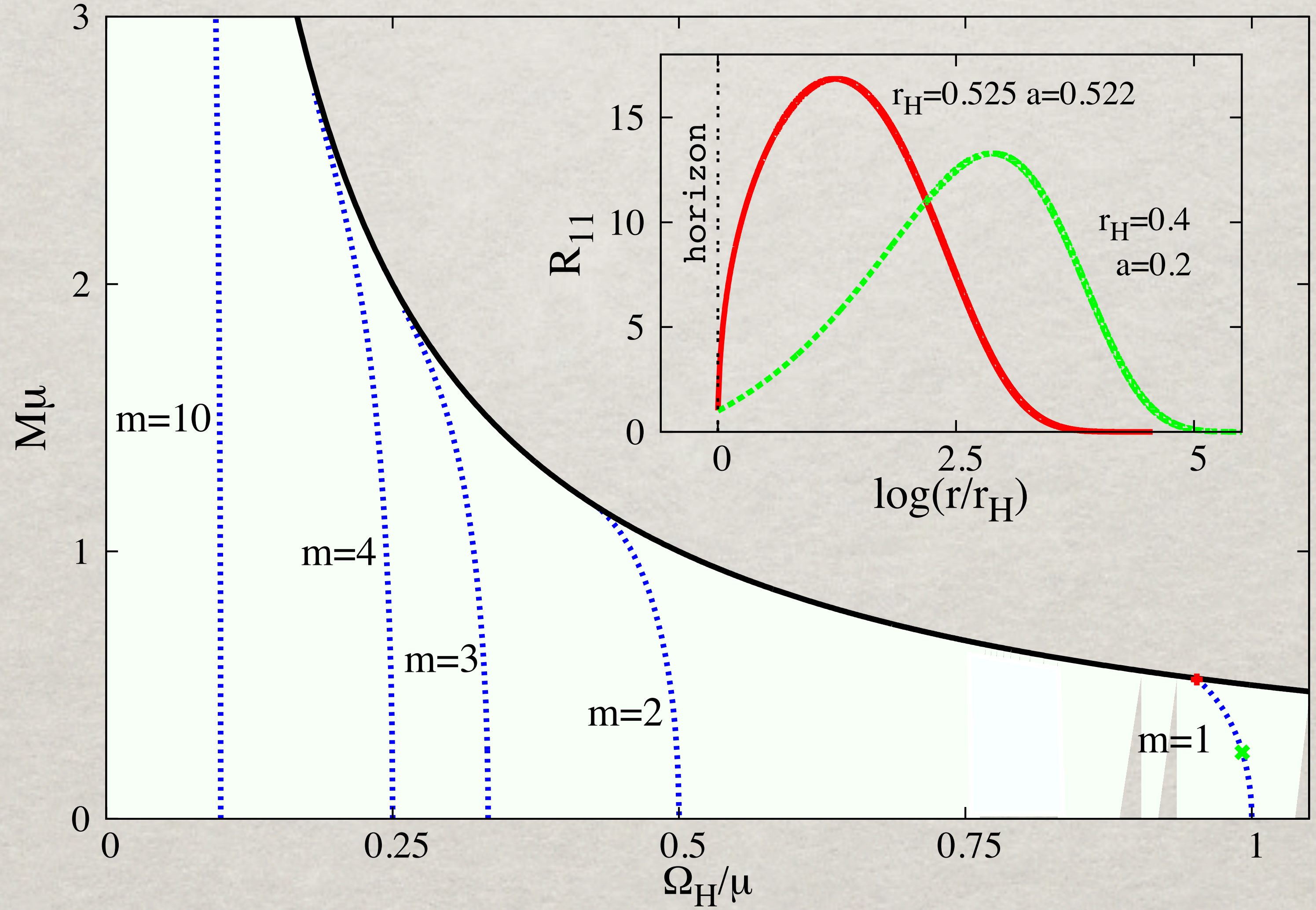
2.4 Scalar stationary clouds on Kerr(-Newman)

By using a numerical technique to solve the radial ODE (say, a shooting method), these **existence points** can be extended to **existence lines** for stationary scalar clouds on the Kerr background CH and Radu, PRL 112 (2014) 221101.



2.4 Scalar stationary clouds on Kerr(-Newman)

Some typical radial profiles of nodeless (n=0) stationary clouds:



2.4 Scalar stationary clouds on Kerr(-Newman)

Question: these stationary **test** clouds are in equilibrium with the black hole. Can we make them **heavy (i.e. backreact)** and get black holes with scalar hair?

Yes: a new type of black holes bifurcates from Kerr

Lessons…

1) The linear field analysis can be repeated for a massive complex vector (Proca) field with similar results.

2) In the case of spontaneous scalarisation, the bifurcation to the scalarised BHs occurs, similarly, at the zero mode of the tachyonic instability.

3) The equality $w/m = \Omega_H$ can be interpreted as synchronisation. In some systems… Nature likes synchrony.

Lecture plan:

a) Black holes have no hair

b) Story I: Linear hair and synchronisation

c) Story II: Non-linear hair - new black holes and solitons

d) Discussion

Stationary scalar solitons in field theory

An interesting question in any gravitational model is whether stationary particle-like solutions exist, i.e. *gravitating solitons*: **everywhere regular configurations, without horizons**, corresponding to localized lumps of (time-independent) gravitational+matter field energy.

There is, however, a generic argument, known as *Derrick's theorem*, against the existence of stable, time-independent solutions of finite energy in a wide class of non-linear wave equations, in three or higher (spatial) dimensions G. H. Derrick, J. Math. Phys. 5 (1964) 1252 (see also R.H. Hobart, Proc. Phys. Soc. 82 (1963)201).

Stationary scalar solitons in field theory

Derrick observed that one way to circumvent the theorem would be allow for localized solutions that are periodic in time, rather than time independent. Various authors, starting with Rosen, considered a complex field with a harmonic time dependence, which guarantees a time-independent energy momentum tensor G. Rosen, J. Math. Phys. 9 (1968) 996:

$$\Phi(t, \mathbf{r}) = e^{-iwt}\varphi(\mathbf{r})$$

Moreover there is a global symmetry and a conserved scalar charge (typically called Q). Then, for some classes of potentials (yielding non-linear models), localized stable solutions exist, which are now known, following Coleman, as *Q-balls* S. R. Coleman, “Q Balls,” Nucl. Phys. B 262 (1985) 263 [Erratum-ibid. B 269 (1986) 744]

But in the presence of gravity, no scalar non-linear interactions are required. Effectively, such non-linearities are provided by the self-gravity of the field.

Gravitating scalar solitons: boson stars

Gravitating scalar solitons: boson stars

The model (mini-boson stars) D. J. Kaup, Phys. Rev. 172 (1968) 1331:

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi} - \frac{1}{2}g^{\alpha\beta}\left(\Phi^*_{,\,\alpha}\Phi_{,\,\beta} + \Phi^*_{,\,\beta}\Phi_{,\,\alpha}\right) - \mu^2\Phi^*\Phi\right],$$

The field equations:

$$G_{\alpha\beta} = 8\pi\left\{\Phi^*_{,\alpha}\Phi_{,\beta} + \Phi^*_{,\beta}\Phi_{,\alpha} - g_{\alpha\beta}\left[\frac{1}{2}g^{\gamma\delta}(\Phi^*_{,\gamma}\Phi_{,\delta} + \Phi^*_{,\delta}\Phi_{,\gamma}) + \mu^2\Phi^*\Phi\right]\right\}$$

$$\Box\Phi = \mu^2\Phi$$

The action is invariant under a U(1) global symmetry: $\Phi \to e^{i\alpha}\Phi$

This leads to a conserved current: $j^\alpha = -i(\Phi^*\partial^\alpha\Phi - \Phi\partial^\alpha\Phi^*)$

Integrating the temporal component of this 4-current on a timelike slice leads to a conserved charge - the *Noether charge* Q:

$$Q = \int_\Sigma j^t$$

The Noether charge counts the number of scalar particles. Notice that this is conserved in the sense of a local continuity equation; **there is no associated Gauss law**!

Gravitating scalar solitons: boson stars

Spherically symmetric solutions ansatz (**three** unknown functions):

$$ds^2 = -N(r)\sigma^2(r)dt^2 + \frac{dr^2}{N(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \quad N(r) \equiv 1 - \frac{2m(r)}{r} \ , \quad \Phi = \phi(r)e^{-iwt}$$

The time dependence cancels at the level of the energy momentum tensor, being therefore compatible with a stationary metric. Thus k = ∂/∂t is a Killing vector field, but it does not preserve the scalar field - the metric and the matter field **do not share the same symmetries**.

The above ansatz makes the Einstein equations simpler as compared to other choices (such as isotropic coordinates). The two "essential" Einstein equations read:

$$m' = 4\pi r^2 \left(N\phi'^2 + \mu^2\phi^2 + \frac{w^2\phi^2}{N\sigma^2}\right) \ , \quad \sigma' = 8\pi\sigma r\left(\phi'^2 + \frac{w^2\phi^2}{N^2\sigma^2}\right)$$

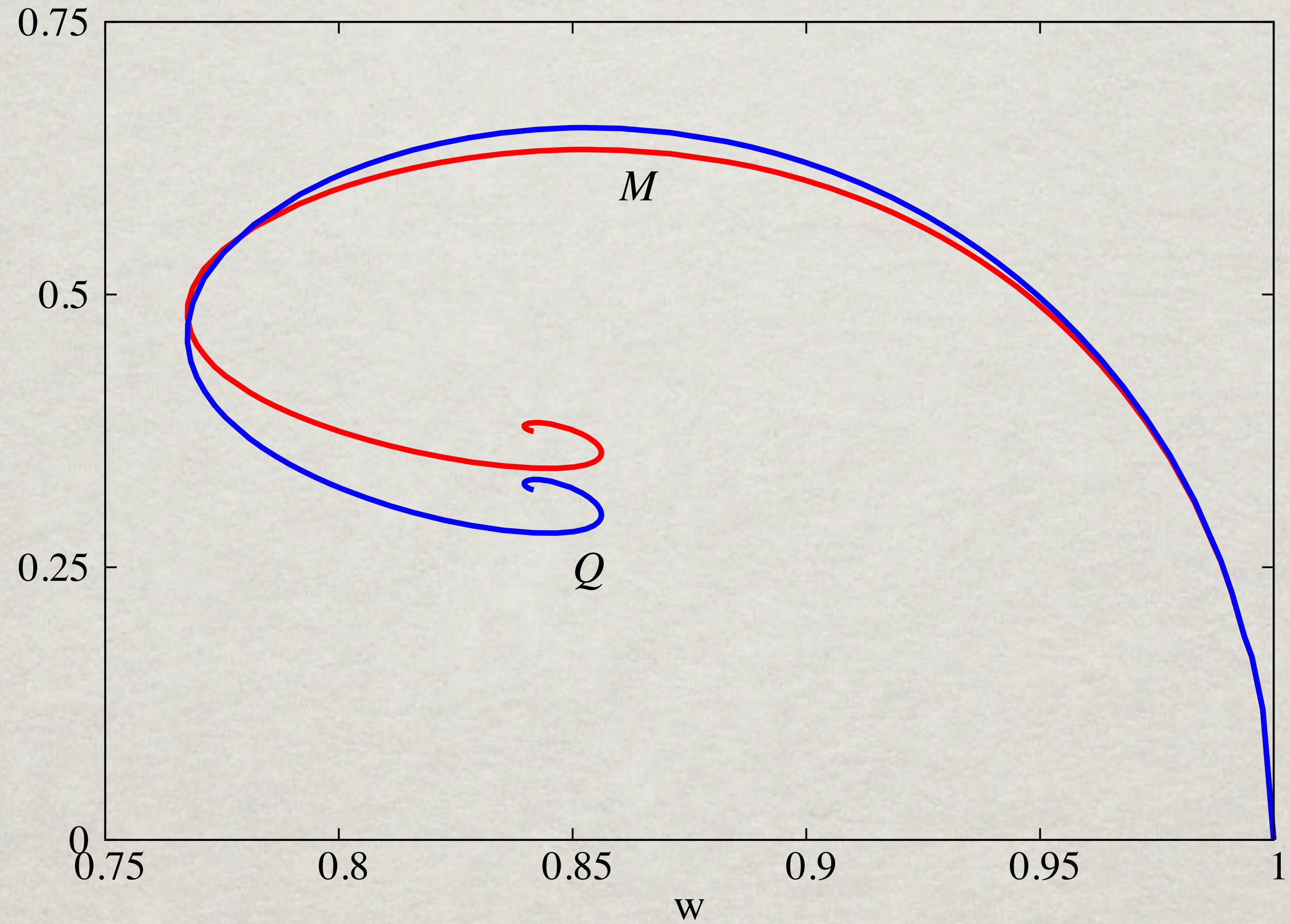
(one further constraint equation is found, but which is a differential consequence of these).

The Klein-Gordon equation gives (thus completing **three** equations):

$$\phi'' + \frac{2\phi'}{r} + \frac{N'\phi'}{N} + \frac{\sigma'\phi'}{\sigma} - \frac{\mu^2\phi}{N} + \frac{w^2\phi}{N^2\sigma^2} = 0$$

Gravitating scalar solitons: boson stars

ADM mass M (and Noether charge Q) vs. frequency w diagram:



In units of μ

$M\mu \to M \ , \quad w/\mu \to w$

- Solutions only exist for a range of frequencies: $\frac{w_{\rm min}}{\mu} < \frac{w}{\mu} < 1$ $\qquad w_{\rm min} \simeq 0.767\mu$

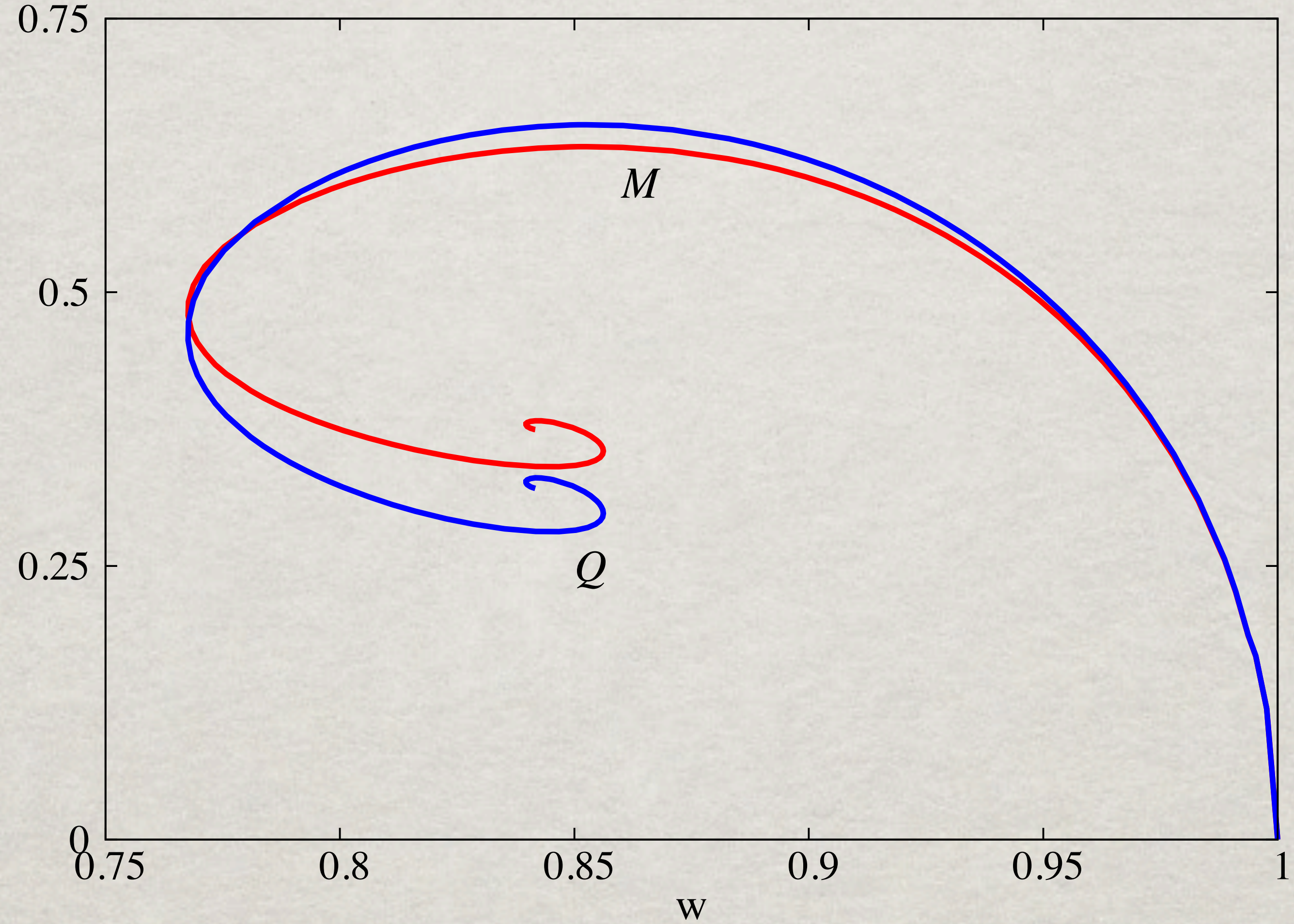
- There is a range of frequencies for which more than one solution exists. This defines the first, second, third, etc, **branches**.

- There is a maximum value for the ADM mass: $M_{\rm ADM}^{\rm max} \simeq \alpha_{\rm BS} \frac{M_{\rm Pl}^2}{\mu} \simeq \alpha_{\rm BS}\, 10^{-19} M_{\odot} \left(\frac{\rm GeV}{\mu}\right)$

$\alpha_{\rm BS} = 0.633$

Gravitating scalar solitons: boson stars

ADM mass M (and Noether charge Q) vs. frequency w diagram:



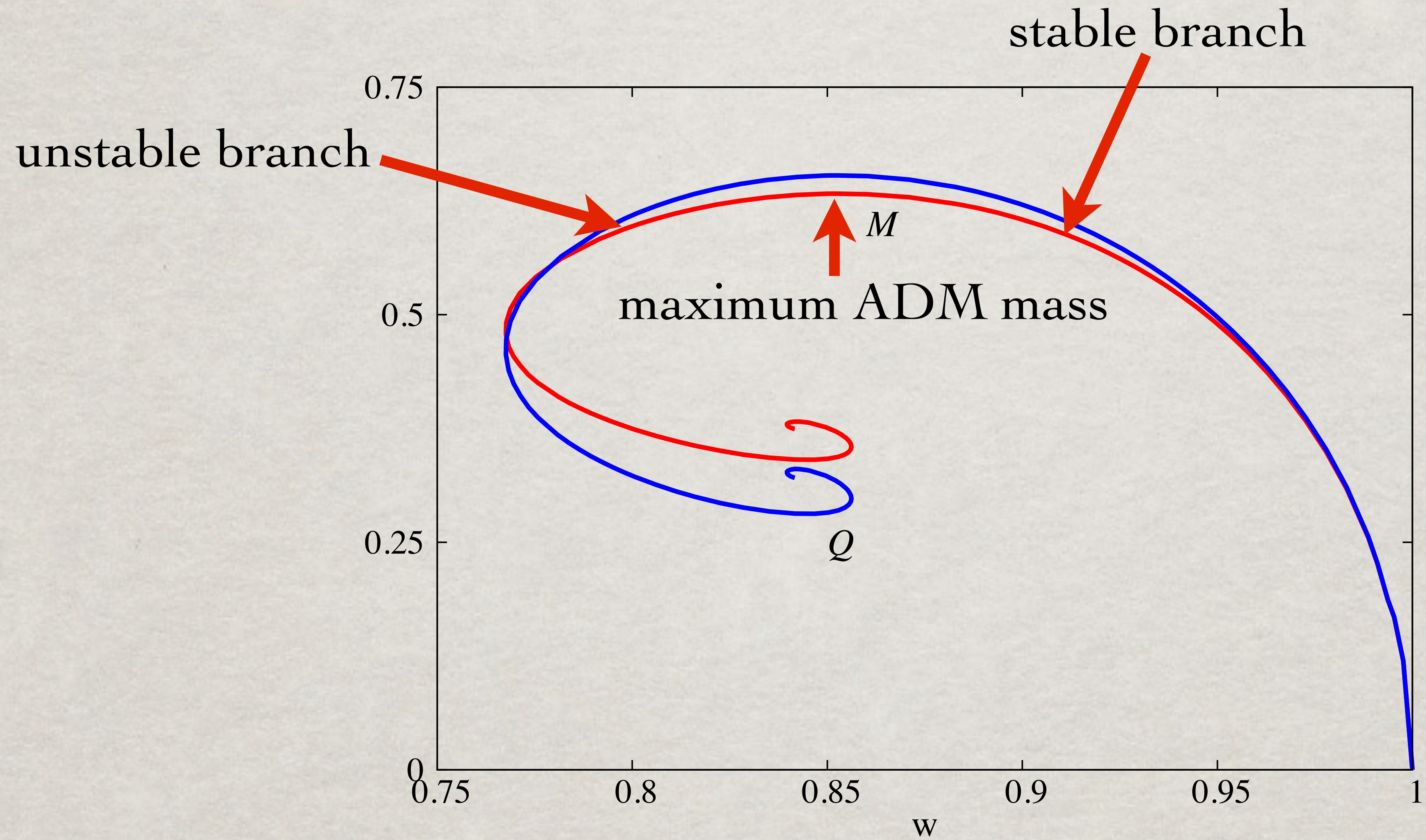
In units of μ

$M\mu \to M \ , \quad w/\mu \to w$

- This spiral corresponds to nodeless solutions. These are regarded as fundamental modes. Excited solutions also exist.

Gravitating scalar solitons: boson stars

Spherically symmetric solutions: Stability



In units of μ

$$M\mu \to M \ , \quad w/\mu \to w$$

Studying linearized radial perturbations of the coupled metric-scalar field system shows that an unstable mode arises precisely at the maximum of the ADM mass M. Gleiser and R. Watkins, Nucl. Phys. B319 (1989) 733; T. D. Lee and Y. Pang, Nucl. Phys. B315, 477 (1989).

Unstable BSs can migrate, decay into a Schwarzschild black hole or disperse entirely Seidel and Suen, PRD 42 (1990) 384; Guzman, PRD 70 (2004) 044033; Hawley and Choptuik, PRD 62 (2000)104024

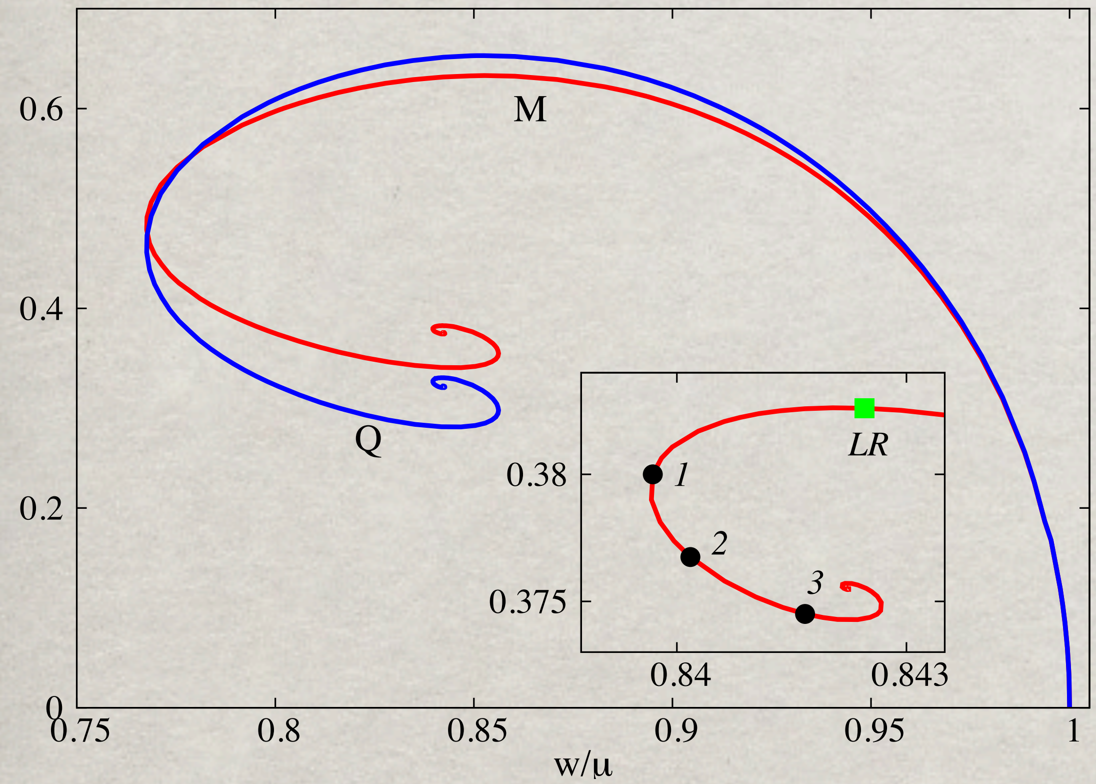
The vector cousin: spherical Proca stars

Brito, Cardoso, Herdeiro and Radu, Phys. Lett. B 752 (2016) 291

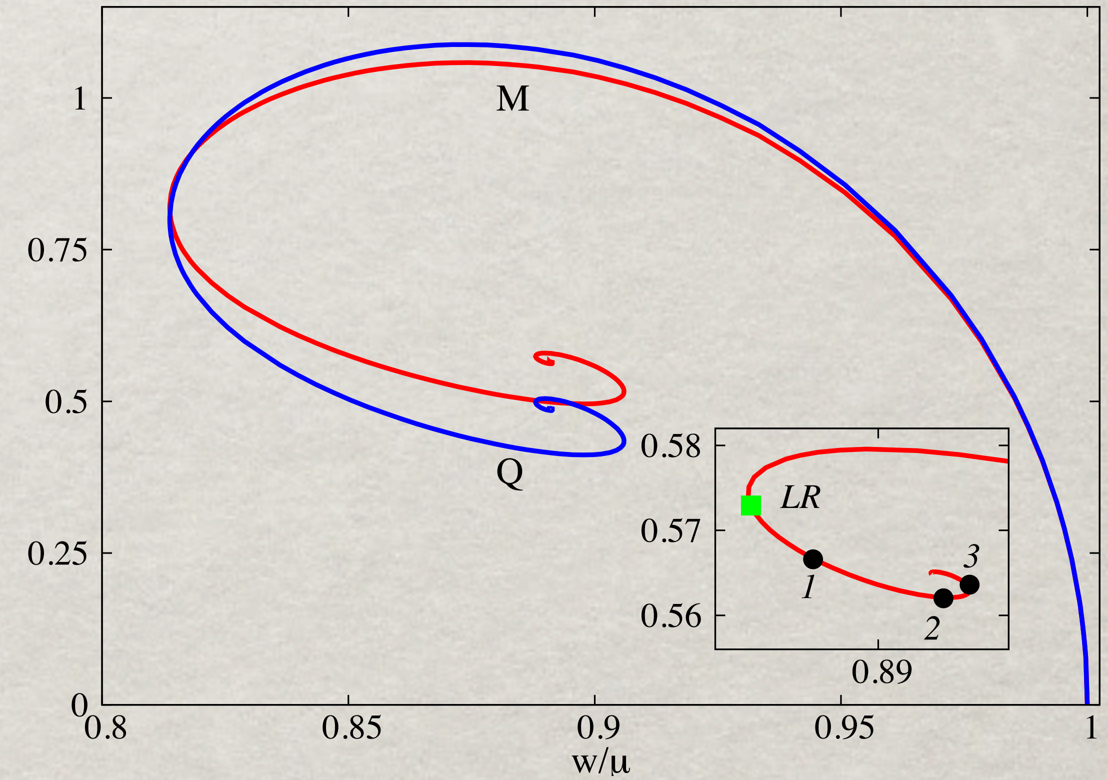
$$\mathcal{S} = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G}R - \frac{1}{4}\mathcal{F}_{\alpha\beta}\bar{\mathcal{F}}^{\alpha\beta} - \frac{1}{2}\mu^2\mathcal{A}_\alpha\bar{\mathcal{A}}^\alpha\right) .$$

A similar construction holds yielding spherical solitonic objects: spherical Proca stars

scalar



vector/Proca



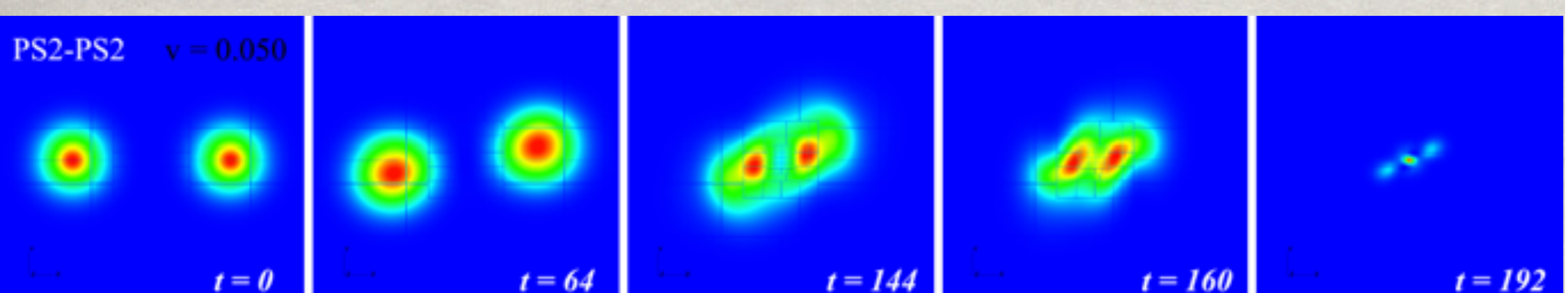
Very similar domain of existence;
Similar structure of fundamental family and excited states.

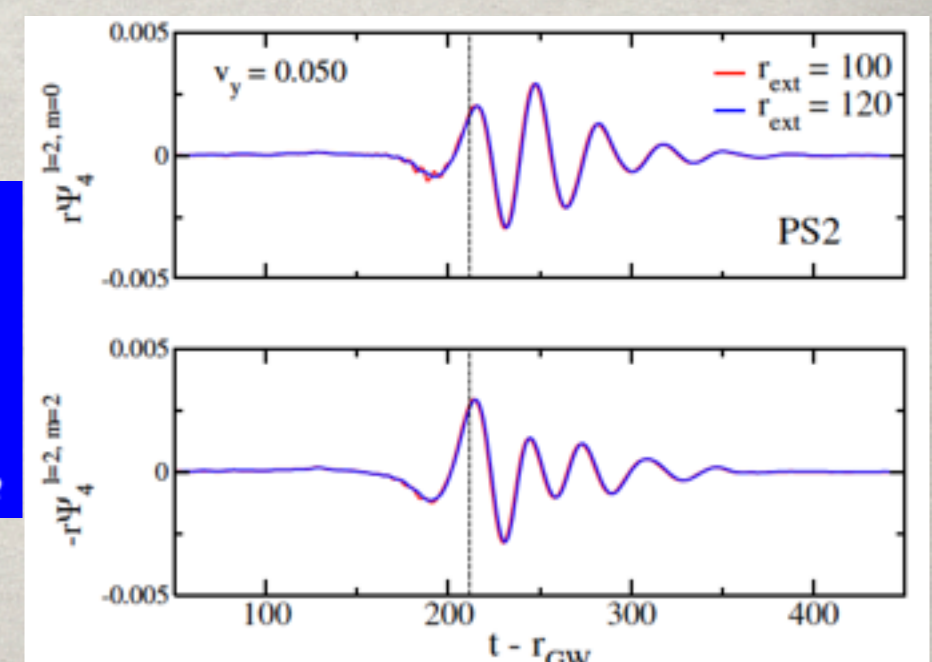
Dynamics of spherical scalar and Proca stars

1) As in the scalar case, vector boson stars are perturbatively stable up to the maximal mass; then they share the same three possible fates: migration, collapse or dispersion
Brito, Cardoso, Herdeiro and Radu, Phys. Lett. B 752 (2016) 291

2) As in the scalar case Seidel and Suen, Phys. Rev. Lett. 72 (1994) 2516, vector boson stars can form dynamically via gravitational cooling
Di Giovanni, Sanchis-Gual, Herdeiro and Font, PRD 98 (2018) 064044

3) As in the scalar case Palenzuela, Pani, Bezares, Cardoso, Lehner and Liebling, Phys. Rev. D 96 (2017) 104058 one can study binaries of spherical Proca stars and their gravitational wave emission
Sanchis-Gual, Herdeiro, Font, Radu and Di Giovanni, Phys. Rev. D 99 (2019) 024017





Stable model; apparent horizon forms at t~200

Rotating boson stars

Axially symmetric solutions ansatz (in quasi-isotropic coordinates) S.Yoshida and Y. Eriguchi, Phys. Rev. D 56 (1997) 762; F. E. Schunck and E. W. Mielke, Phys. Lett. A 249 (1998) 389:

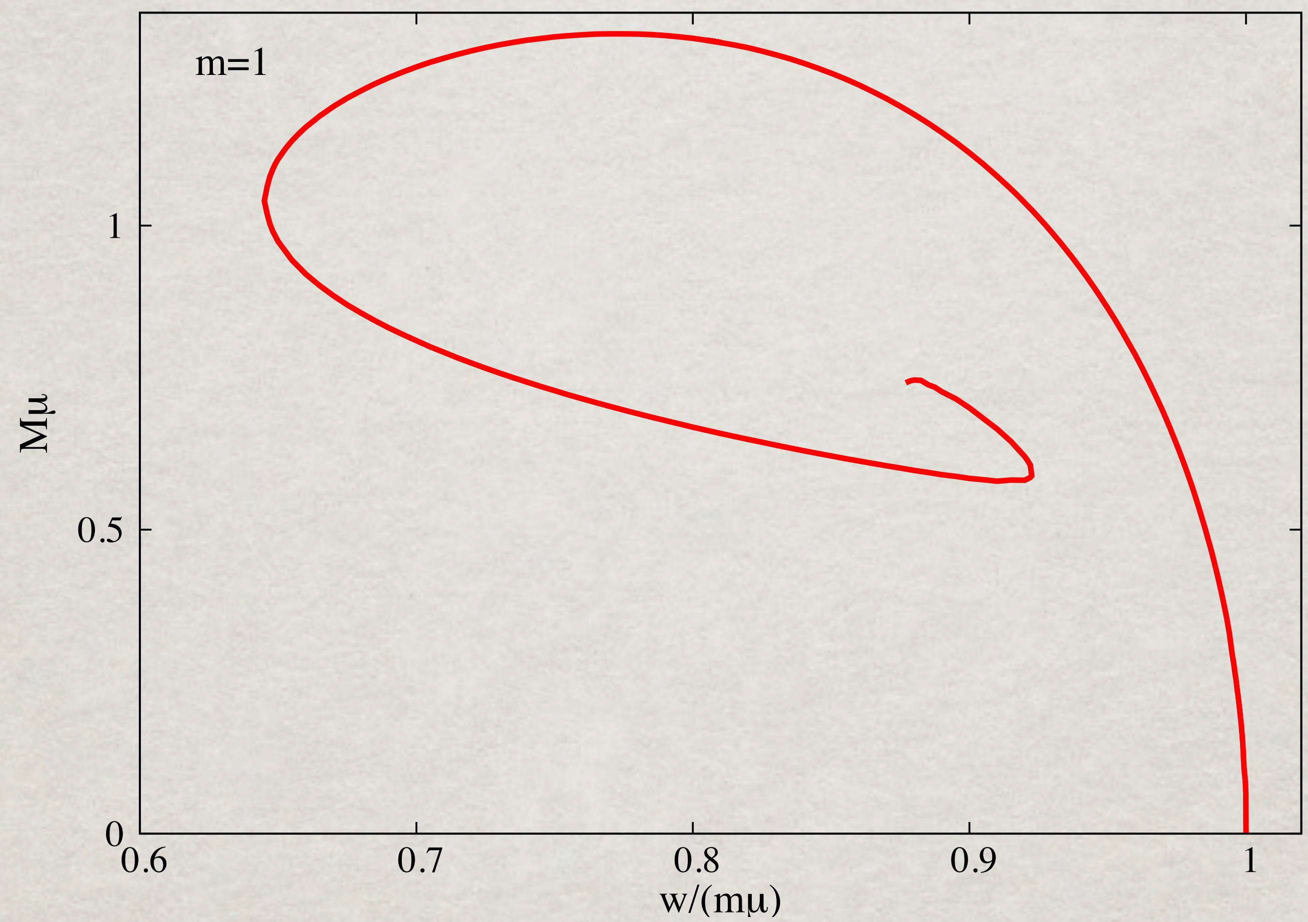
$$ds^2 = -e^{2F_0(r,\theta)}dt^2 + e^{2F_1(r,\theta)}\left(dr^2 + r^2 d\theta^2\right) + e^{2F_2(r,\theta)}r^2\sin^2\theta\left(d\varphi - W(r,\theta)dt\right)^2 \qquad \Phi = \phi(r,\theta)e^{i(m\varphi - wt)}$$

The solution has three parameters: (w,m,n), but again these do not define solutions uniquely.

Solutions preserved by a single helicoidal Killing vector field:

$$\frac{\partial}{\partial t} + \frac{w}{m}\frac{\partial}{\partial \varphi}$$

Rotating boson stars



The maximum value for the ADM mass increases with m:

$$M_{\rm ADM}^{\rm max} \simeq \alpha_{\rm BS} \frac{M_{\rm Pl}^2}{\mu} \simeq \alpha_{\rm BS}\, 10^{-19} M_\odot \left(\frac{\rm GeV}{\mu}\right)$$

m=0: $\alpha_{\rm BS} = 0.633$

m=1: $\alpha_{\rm BS} = 1.315$

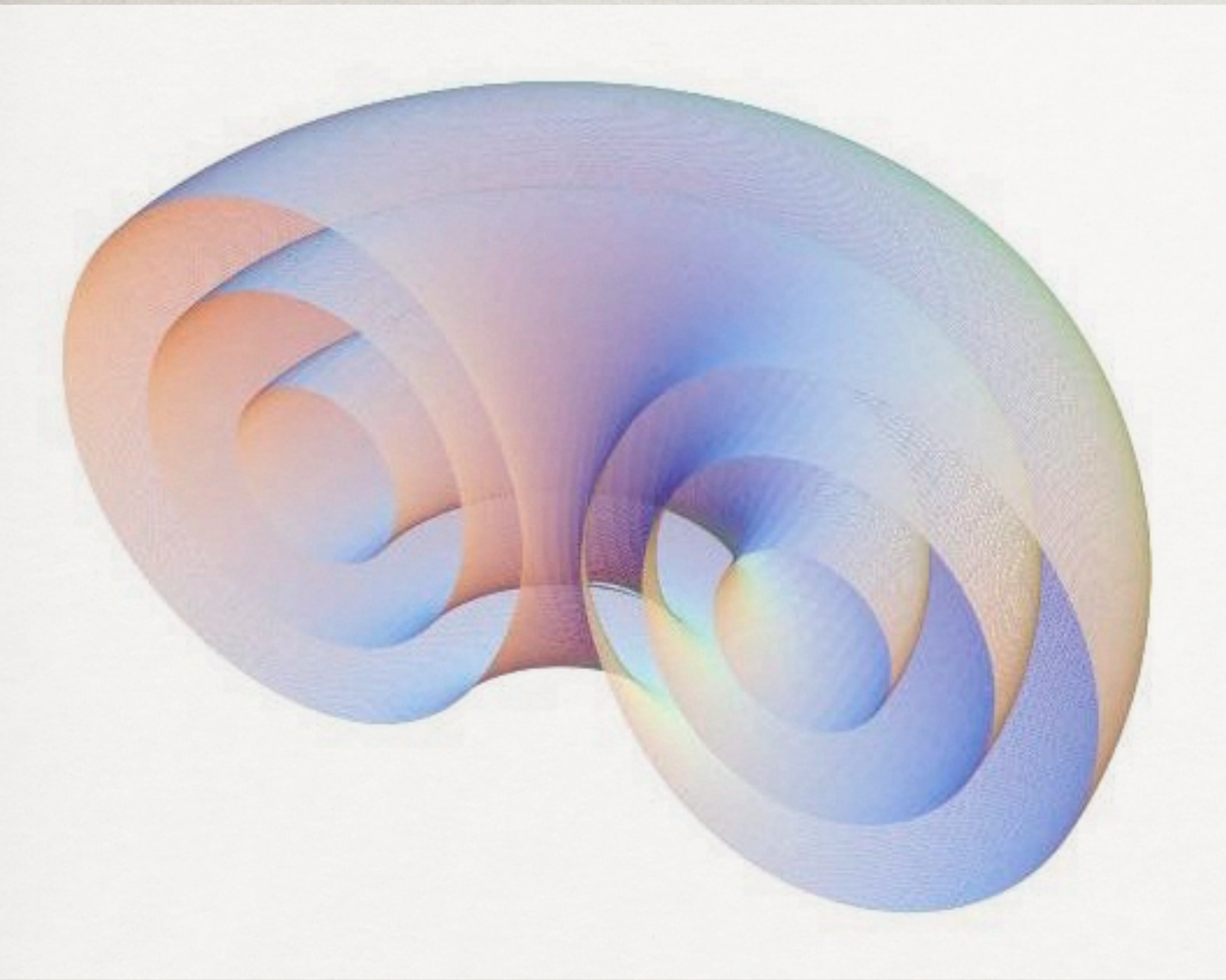
S.Yoshida and Y. Eriguchi, Phys. Rev. D 56 (1997) 762

m=2: $\alpha_{\rm BS} = 2.216$

P. Grandclement, C. Somé and E. Gourgoulhon, Phys. Rev. D 90 (2014) 2, 024068 [arXiv:1405.4837 [gr-qc]].

For a vector model this leads to rotating Proca stars

Brito, Cardoso, CH and Radu, PLB 752 (2016) 291

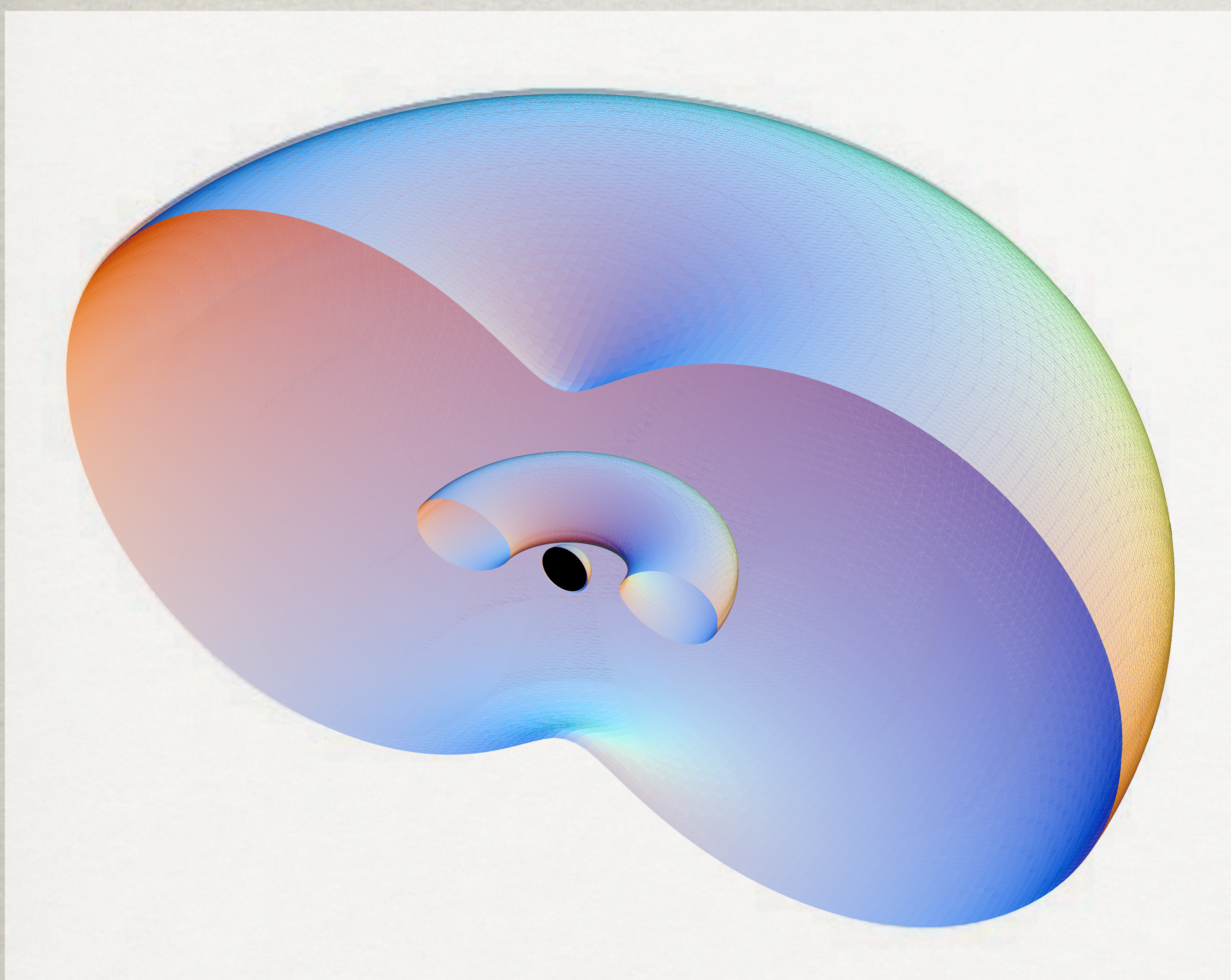


Rotating scalar boson stars

Rotating Proca stars

In both cases the construction of the
solitonic objects can be generalised to
hairy black holes, imposing the synchronisation condition.

Circumvent many various no hair theorem due to the matter field not inheriting the metric isometries



BHs with synchronised scalar hair

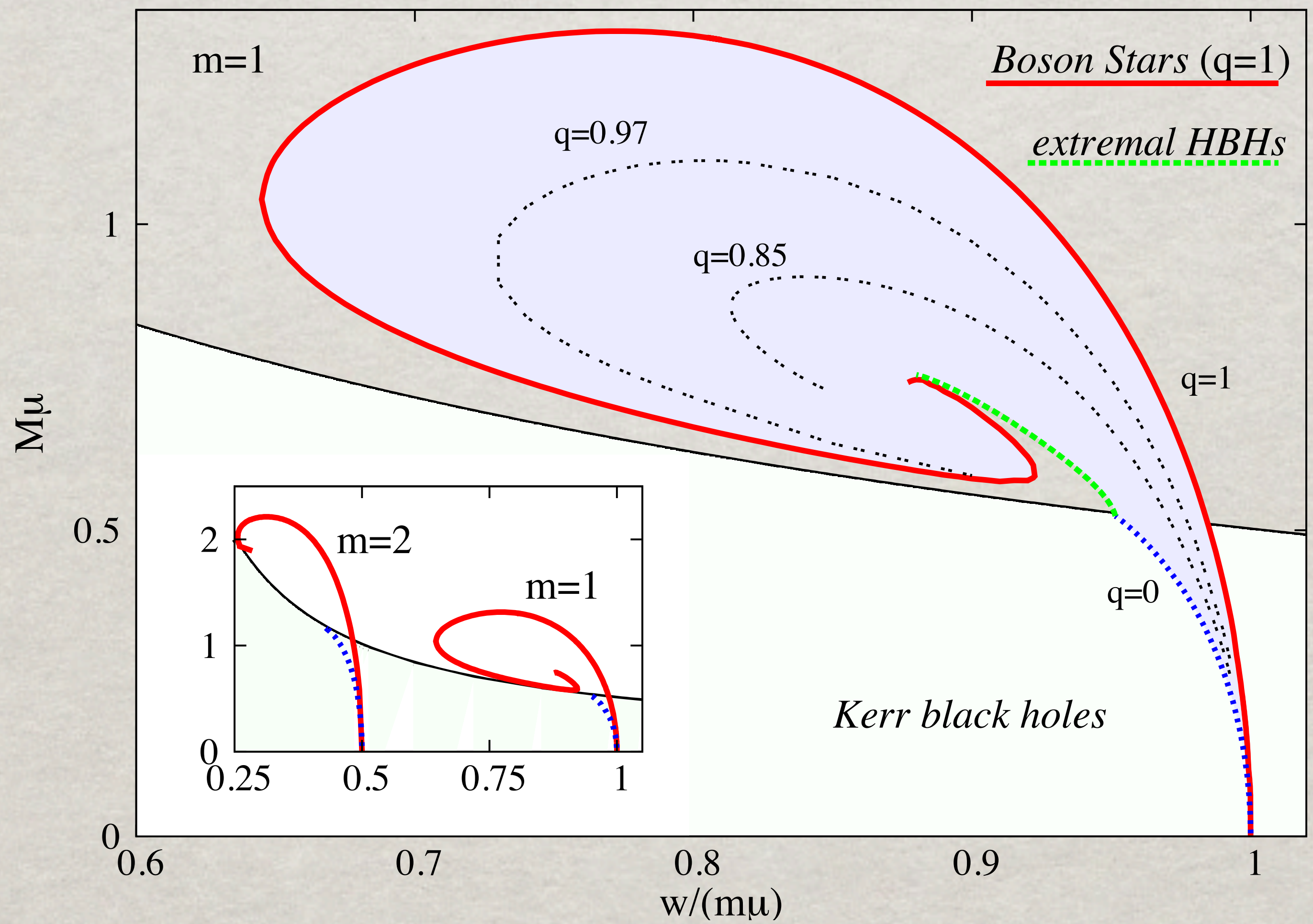
CH and Radu, Phys. Rev. Lett. 112 (2014) 221101

BHs with synchronised Proca hair

CH, Radu and Rúnarsson, Class. Quant. Grav. 33 (2016) 154001

Hairy black holes are the non-linear realization of the stationary clouds.

Their family interpolates between Kerr black holes
and
the corresponding bosonic (scalar or vector) stars



$$q \equiv \frac{mQ}{J}$$

Lesson...

The e^{-iwt} ingredient

and

synchronisation

provide a generic

mechanism to endow rotating black holes with

hair of a fundamental field.

Lecture plan:

a) Introduction: the simplicity of black holes

b) Story I: Linear analysis and new dof (“hair”)

c) Story II: Non-linear analysis - new black holes and solitons

d) Discussion

Black holes in the simplest GR models

$$\mathcal{S} = \frac{1}{16\pi}\int d^4x\sqrt{-g}R$$

Vacuum

Kerr

$+A_\mu$

Electro-vacuum

Kerr-Newman

$+\Phi$

plus massive complex scalar

Synchronised hairy

Existence proof

Chodosh and Shlapentokh-Rothman, CMP356(2017)1155

Reasonable ?

Phenomenology?

“Reasonable” non-Kerr black holes:

Theoretical criteria:

1) Appear in a well motivated and consistent physical model;

Kerr: General Relativity

2) Have a dynamical formation mechanism;

Kerr: gravitational collapse

3) Be (sufficiently) stable.

Kerr: mode stability established (B. F. Whiting, J. Math. Phys. 30 (1989) 1301)

Correct phenomenology:

1) all electromagnetic observables
(X-ray spectrum, shadows, QPOs, star orbits,...);

2) correct Gravitational wave templates

No clear tension between observations and the Kerr model

Q&A

Q: is there a mechanism of formation for the black holes with synchronised hair?

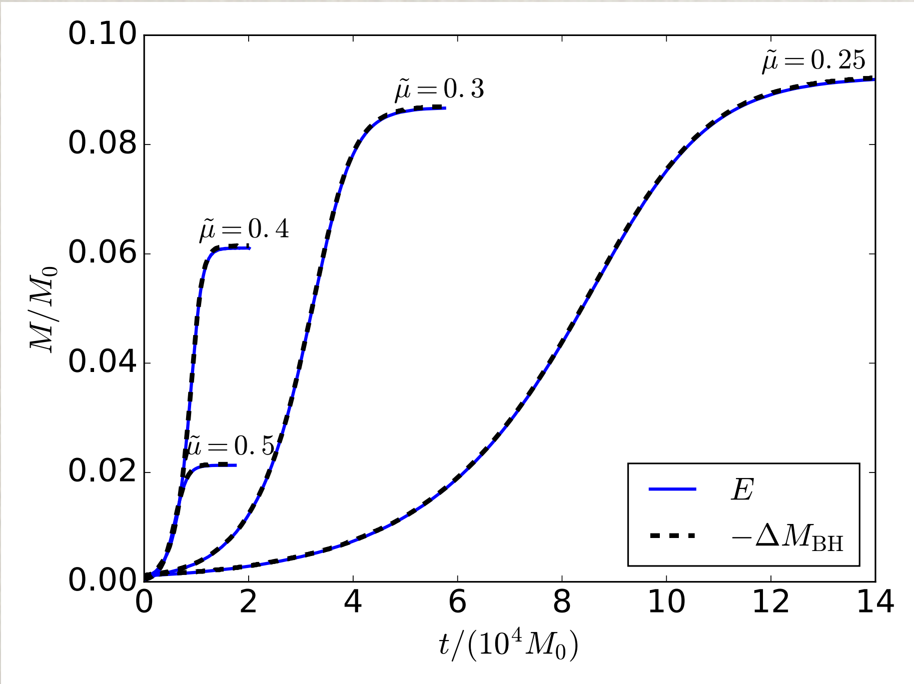
A: Yes. The superradiant instability of the Kerr BH in the presence of an ultralight bosonic field.

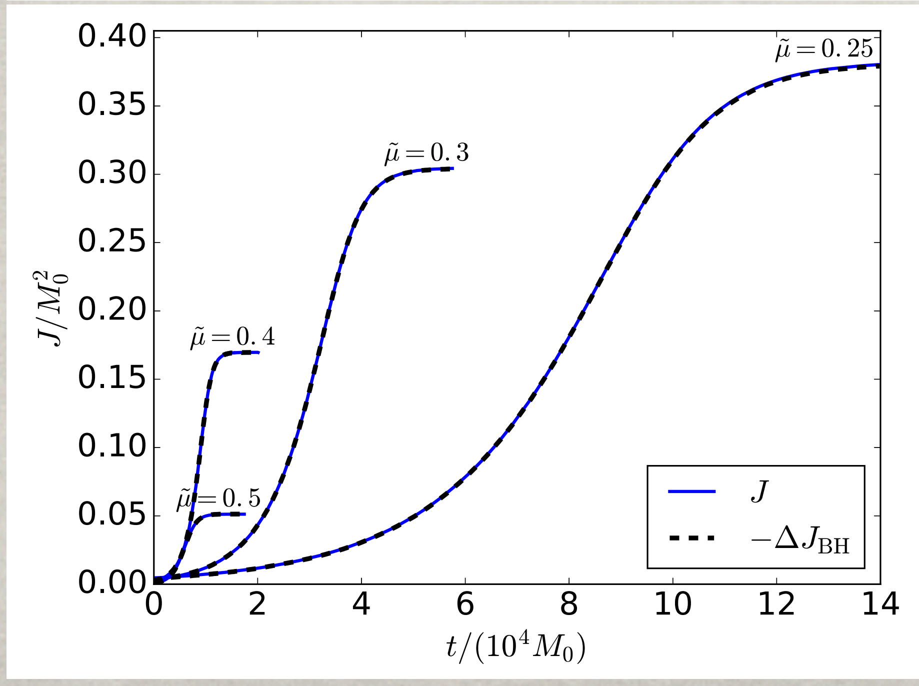
East and Pretorius, PRL119(2017)041101
CH, Radu, PRL 119 (2017) 261101
Dolan, Physics10(2017)83

Dynamical evidence (for the cousin Proca model) shows the process reaches an equilibrium state...

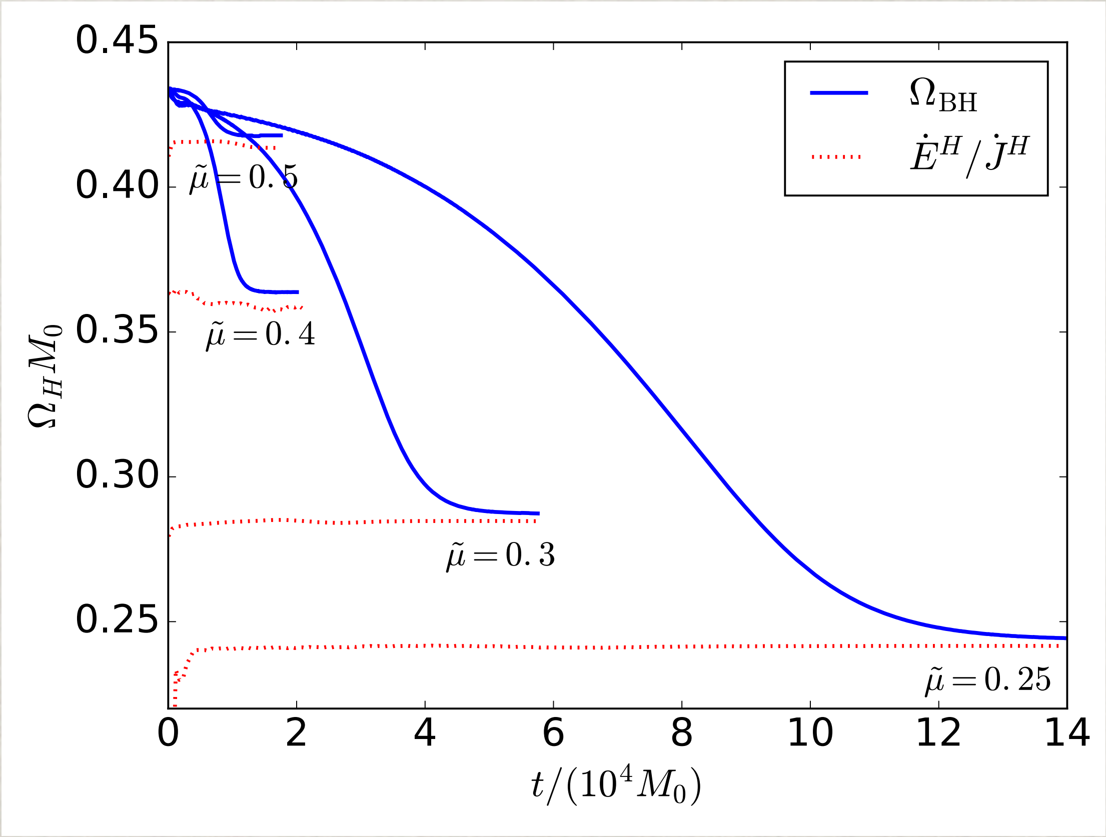
East and Pretorius, PRL119(2017)041101

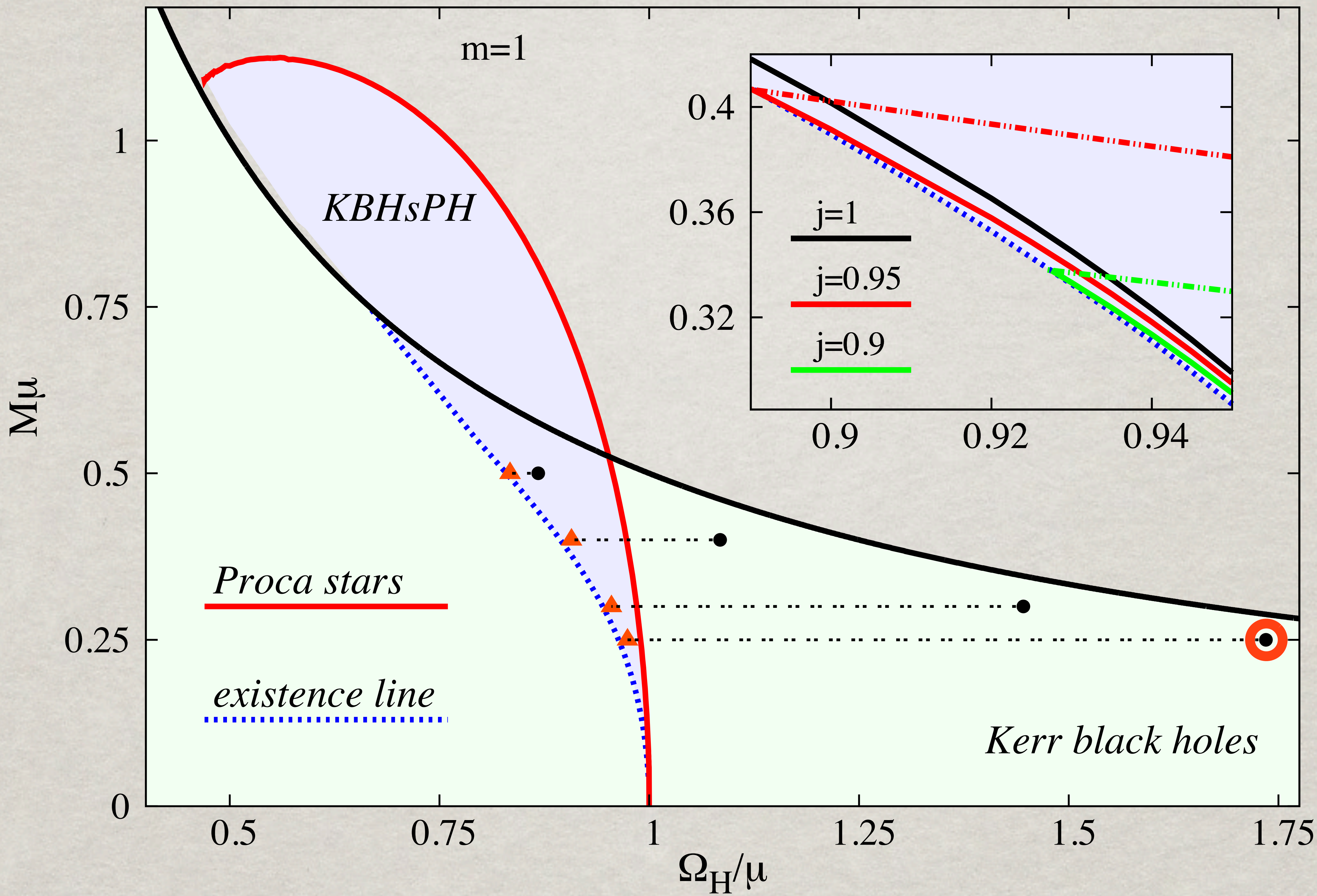
Mass and angular momentum in "hair"





Black hole spin down and "synchronisation"

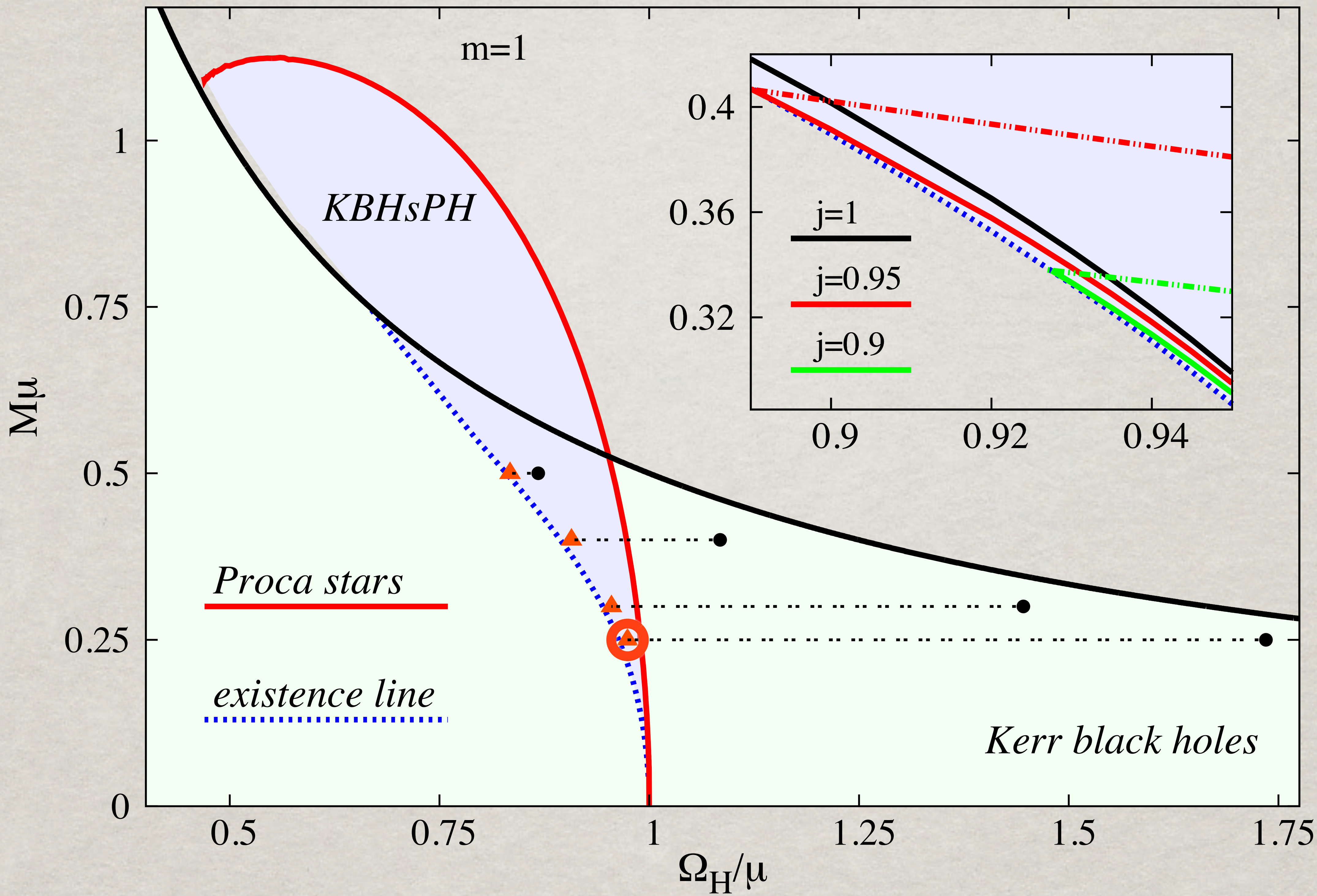




... which is a hairy black hole

CH, Radu, PRL 119 (2017) 261101

Dolan, Physics10(2017)83



... which is a hairy black hole

CH, Radu, Phys. Rev. Lett. 119 (2017) 261101

Dolan, Physics10(2017)83

Q&A

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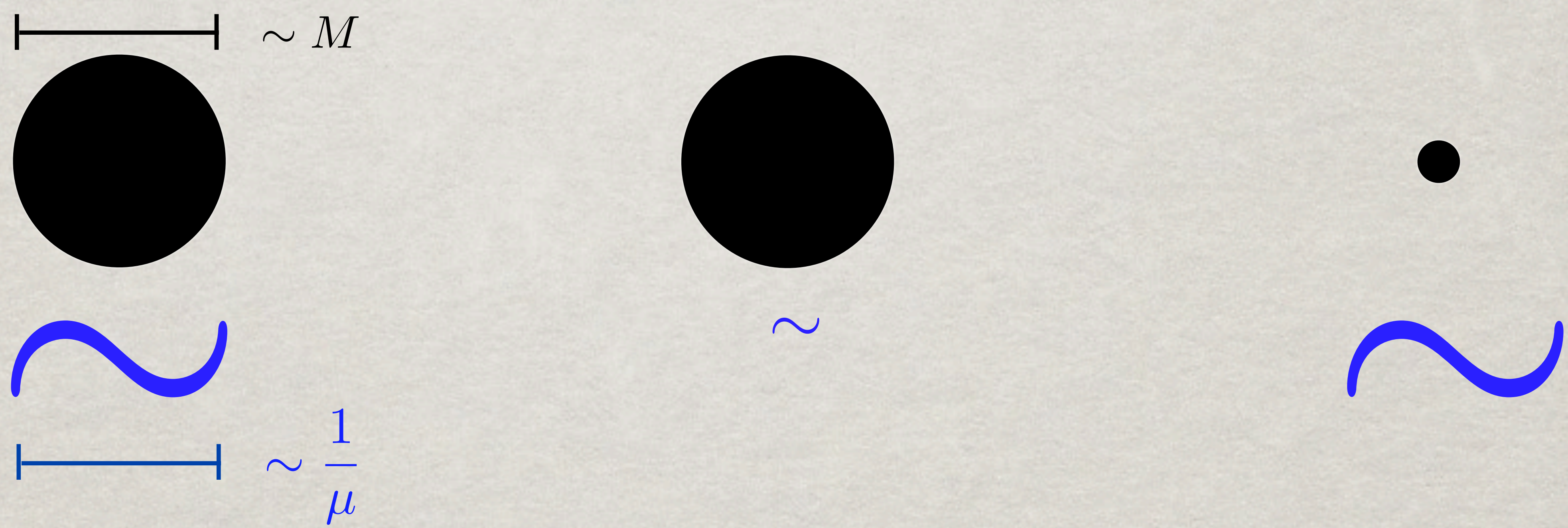
East and Pretorius, PRL119(2017)041101
CH, Radu, PRL 119 (2017) 261101
Dolan, Physics10(2017)83

Note: The superradiant instability is very sensitive to a resonance between the Compton wavelength of the particle and the gravitational scale of the black hole -> Selects a scale where black holes grow hair.

$$\text{Kerr} \xrightarrow{\Delta t} \text{Hairy black hole}$$

Time scale depends on:

1) μM



Maximal efficiency $\mu M \sim 1$

Exponential increase $\mu M \gg 1$

(high) power law increase $\mu M \ll 1$

2) Black hole spin: most efficient for (almost) extremal black holes

Q&A

Q: are the black holes with synchronised hair absolutely stable?

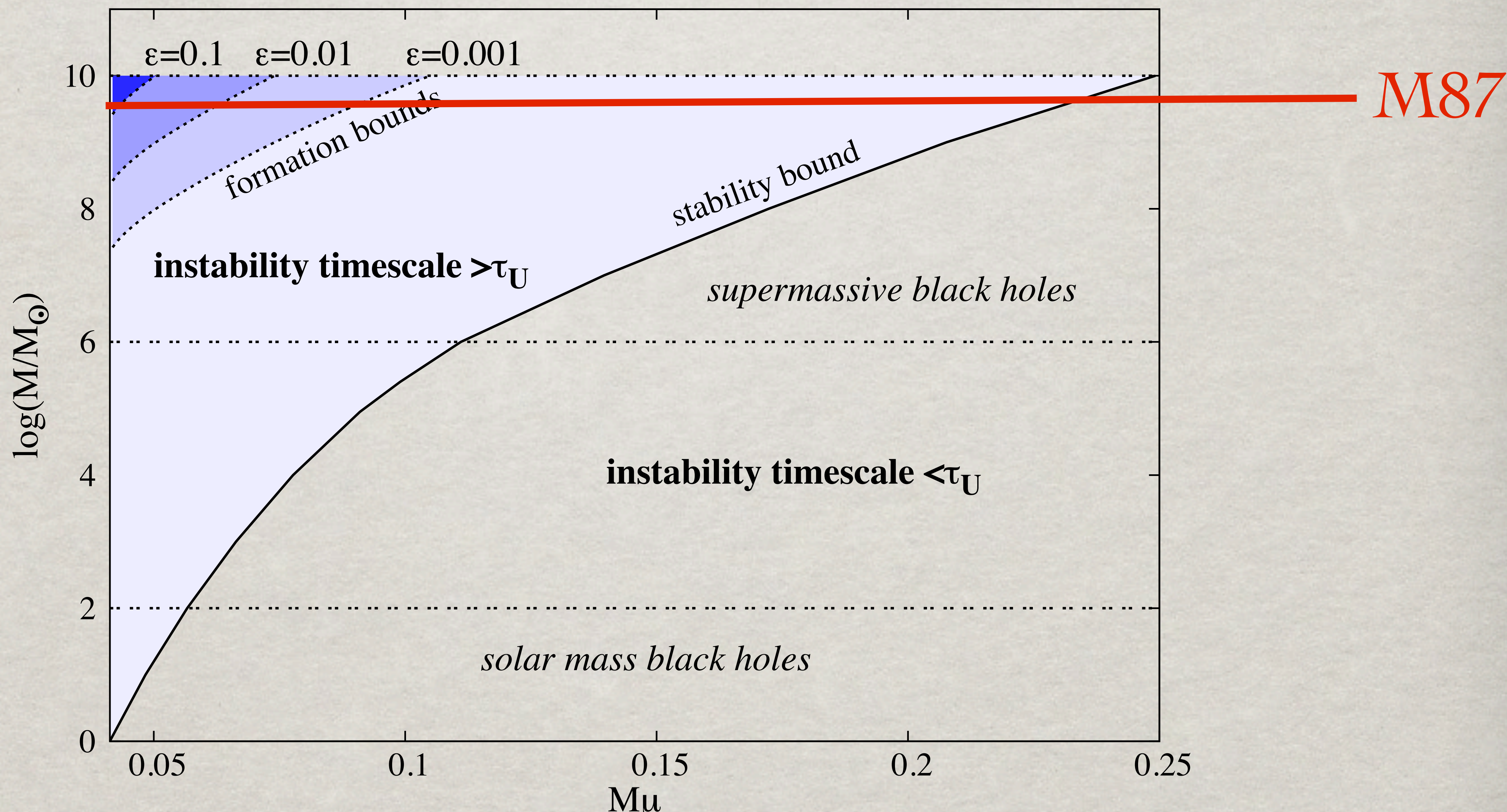
A: No. They are themselves prone to superradiant instabilities of higher modes.

CH, Radu, PRD 89 (2014) 124018
Ganchev and Santos PRL 120 (2018) 171101
Degollado, CH, Radu PLB 781 (2018) 651

Note: There are hairy BHs for which the instability timescale is larger than the age of the Universe: *effective stability*.

A conservative estimate of the Astrophysically viable region

Degollado, CH, Radu PLB 781 (2018) 651



EHT constraints on the amount of synchronised hair

Cunha, CH and Radu, arXiv:1909.08039 [gr-qc]

Q&A

Q: is the phenomenology of these black holes known?

A: Some, related to electromagnetic observables (X-ray spectrum, shadows, QPOs,...).

Black holes with little hair are Kerr-like; very hairy black holes are Kerr-unlike.

Missing dynamical studies (both perturbative and fully non-linear) to assess gravitational wave physics.

But started recently studies of the dynamics or rotating scalar and vector boson stars - some surprises Sanchis-Gual, Di Giovanni, Zilhão, Herdeiro, P. Cerda-Duran, Font and Radu, arXiv:1907.12565

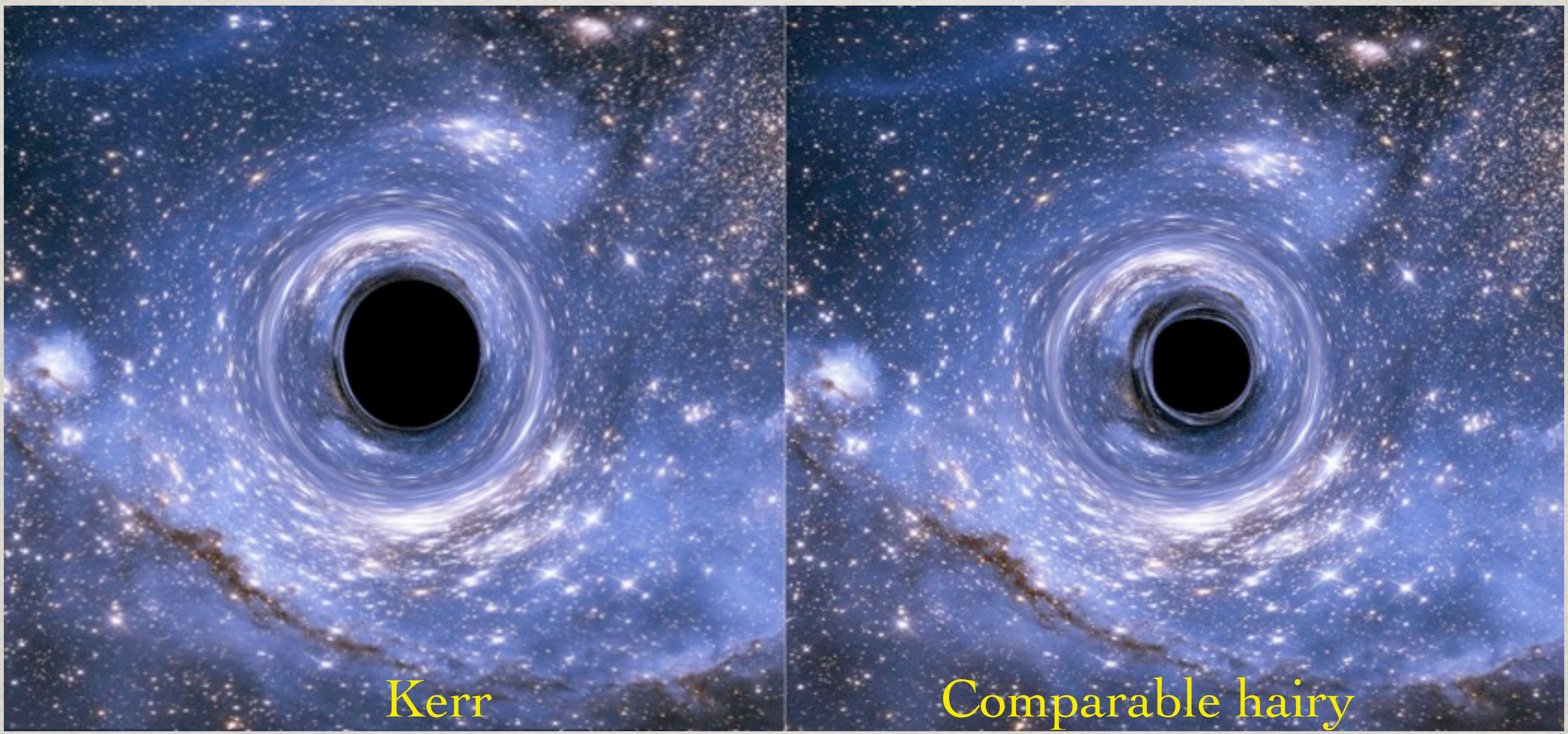
Lensing of the

Infant Stars in Small Magellanic cloud (HST)

by a black hole with scalar synchronised hair

Cunha, CH, Radu, Runarsson,

Phys. Rev. Lett. 115 (2015) 211102



Thank you for your attention!