Non-linear Boltzmann Equations for the Cosmic Microwave Background

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COSMO 2013 at Cambridge

Second-order Line of Sight Approach

$$\hat{\mathcal{I}}^{[\mathrm{II}]}(\eta_0, \mathbf{k}, \hat{\mathbf{n}}) = \int_0^{\eta_0} \mathrm{d}r \; e^{-i\mathbf{k}\cdot\hat{\mathbf{n}}r - \bar{\tau}(\eta)} \int \frac{\mathrm{d}\mathbf{k}_1 \mathrm{d}\mathbf{k}_2}{(2\pi)^{\frac{3}{2}}} \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \hat{\mathcal{S}}_T^{[\mathrm{II}]}(\eta, \mathbf{k}_1, \mathbf{k}_1, \hat{\mathbf{n}})$$

$$\mathcal{S}_T^{[\mathrm{II}]}(\eta, \mathbf{x}, \hat{\mathbf{n}}, p^0) = \mathfrak{C}^{[\mathrm{II}]} + \bar{\tau}' \mathcal{I}^{[\mathrm{II}]} - 2\left(\frac{\mathrm{d}p^0}{\mathrm{d}\eta}\right)^{[\mathrm{I}]} \frac{\partial \mathcal{I}^{[\mathrm{I}]}}{\partial p^0} - \left(\frac{\mathrm{d}p^0}{\mathrm{d}\eta}\right)^{[\mathrm{II}]} \frac{\partial \bar{\mathcal{I}}}{\partial p^0} - 2\left(\frac{\mathrm{d}x^I}{\mathrm{d}\eta}\right)^{[\mathrm{I}]} \frac{\partial \mathcal{I}^{[\mathrm{I}]}}{\partial x^I} - 2\left(\frac{\mathrm{d}n^i}{\mathrm{d}\eta}\right)^{[\mathrm{I}]} \frac{\partial \mathcal{I}^{[\mathrm{I}]}}{\partial n^i}$$

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Compton scattering

Second-order Line of Sight Approach

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Compton scattering

2^{nd}-order redshifts

Photon-redshift coupling

Second-order Line of Sight Approach

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- $\mathfrak{C}^{[\mathrm{II}]} + \bar{\tau}' \mathcal{I}^{[\mathrm{II}]}$: Compton scattering
- $2\left(\frac{\mathrm{d}p^0}{\mathrm{d}\eta}\right)^{[\mathrm{I}]} \frac{\partial \mathcal{I}^{[\mathrm{I}]}}{\partial p^0}$: Photon-redshift coupling
- $\left(\frac{\mathrm{d}p^0}{\mathrm{d}\eta}\right)^{[\mathrm{II}]} \frac{\partial \bar{\mathcal{I}}}{\partial p^0}$: 2^{nd}-order redshifts
- $2\left(\frac{\mathrm{d}x^I}{\mathrm{d}\eta}\right)^{[\mathrm{I}]} \frac{\partial \mathcal{I}^{[\mathrm{I}]}}{\partial x^I}$: Time-delay
- $2\left(\frac{\mathrm{d}n^i}{\mathrm{d}\eta}\right)^{[\mathrm{I}]} \frac{\partial \mathcal{I}^{[\mathrm{I}]}}{\partial n^i}$: Lensing

Second-order Line of Sight Approach

$$\hat{\mathcal{I}}^{[\mathrm{II}]}(\eta_0,\mathbf{k},\hat{\mathbf{n}}) = \int_0^{\eta_0} \mathrm{d}r\; e^{-i\mathbf{k}\cdot\hat{\mathbf{n}}r-\bar{\tau}(\eta)} \int \frac{\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2}{(2\pi)^{\frac{3}{2}}}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\hat{\mathcal{S}}_T^{[\mathrm{II}]}(\eta,\mathbf{k}_1,\mathbf{k}_1,\hat{\mathbf{n}})$$

No lensing and time-delay

$$\mathcal{S}_T^{[\mathrm{II}]}(\eta,\mathbf{x},\hat{\mathbf{n}},p^0) = \mathfrak{C}^{[\mathrm{II}]} + \bar{\tau}'\mathcal{I}^{[\mathrm{II}]} - 2\left(\frac{\mathrm{d}p^0}{\mathrm{d}\eta}\right)^{[\mathrm{I}]}\frac{\partial\mathcal{I}^{[\mathrm{I}]}}{\partial p^0} - \left(\frac{\mathrm{d}p^0}{\mathrm{d}\eta}\right)^{[\mathrm{II}]}\frac{\partial\bar{\mathcal{I}}}{\partial p^0} - 2\left(\frac{\mathrm{d}x^I}{\mathrm{d}\eta}\right)^{[\mathrm{I}]}\frac{\partial\mathcal{I}^{[\mathrm{I}]}}{\partial x^I} - 2\left(\frac{\mathrm{d}n^i}{\mathrm{d}\eta}\right)^{[\mathrm{I}]}\frac{\partial\mathcal{I}^{[\mathrm{I}]}}{\partial n^i}$$

Compton scattering

Photon-redshift coupling

2^{nd}-order redshifts

Time-delay

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$$\mathcal{S}_T^{[\mathrm{II}]}(\eta, \mathbf{x}, \hat{\mathbf{n}}, p^0) = \boxed{\mathfrak{C}^{[\mathrm{II}]} + \bar{\tau}'\mathcal{I}^{[\mathrm{II}]}} - \boxed{2\left(\frac{\mathrm{d}p^0}{\mathrm{d}\eta}\right)^{[\mathrm{I}]} \frac{\partial \mathcal{I}^{[\mathrm{I}]}}{\partial p^0}} - \boxed{\left(\frac{\mathrm{d}p^0}{\mathrm{d}\eta}\right)^{[\mathrm{II}]} \frac{\partial \bar{\mathcal{I}}}{\partial p^0}} - \boxed{2\left(\frac{\mathrm{d}x^I}{\mathrm{d}\eta}\right)^{[\mathrm{I}]} \frac{\partial \mathcal{I}^{[\mathrm{I}]}}{\partial x^I}} - \boxed{2\left(\frac{\mathrm{d}n^i}{\mathrm{d}\eta}\right)^{[\mathrm{I}]} \frac{\partial \mathcal{I}^{[\mathrm{I}]}}{\partial n^i}}$$

Compton scattering

Photon-redshift coupling

2^{nd}-order redshifts

Time-delay

Lensing

Photon-Redshift Coupling

$$\mathcal{S}_T^{[\mathrm{II}]} = -2\left(\frac{\mathrm{d}p^0}{\mathrm{d}\eta}\right)^{[\mathrm{I}]}\frac{\partial \mathcal{I}^{[\mathrm{I}]}}{\partial p^0} = -2\ p^0(\partial_\eta \Psi^{[\mathrm{I}]} - n^i \partial_I \Phi^{[\mathrm{I}]})\frac{\partial \mathcal{I}^{[\mathrm{I}]}}{\partial p^0}$$

Photon-Redshift Coupling

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- Numerically unstable
  - Doesn't converge at low l's $\mathcal{S}_T^{[\mathrm{II}]}(\mathbf{k}_1, \mathbf{k}_2, \hat{\mathbf{n}}, r) = \sum_{lm} \mathcal{S}_{lm}^{[\mathrm{II}]}(\mathbf{k}_1, \mathbf{k}_2, r) Y_{\hat{\mathbf{k}}}^{lm}(\hat{\mathbf{n}})$

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- Our approach: Integrating by parts

$$\hat{\mathcal{I}}^{[\mathrm{II}]} \sim \int_0^{\eta_0} \mathrm{d}\tilde{\eta} e^{-i\mathbf{k}_1 \cdot \hat{\mathbf{n}}\tilde{r}} \mathcal{S}_T^{[\mathrm{I}]}(\tilde{\eta}, \mathbf{k}_1, \hat{\mathbf{n}}) \left\{ e^{-i\mathbf{k}_2 \cdot \hat{\mathbf{n}}\tilde{r}} \Phi(k_2, \tilde{\eta}) + \int_{\tilde{\eta}}^{\eta_0} \mathrm{d}\eta_1 e^{-i\mathbf{k}_2 \cdot \hat{\mathbf{n}} r_1} \left[\Phi'(k_2, \eta_1) + \Psi'(k_2, \eta_1)\right] \right\}$$

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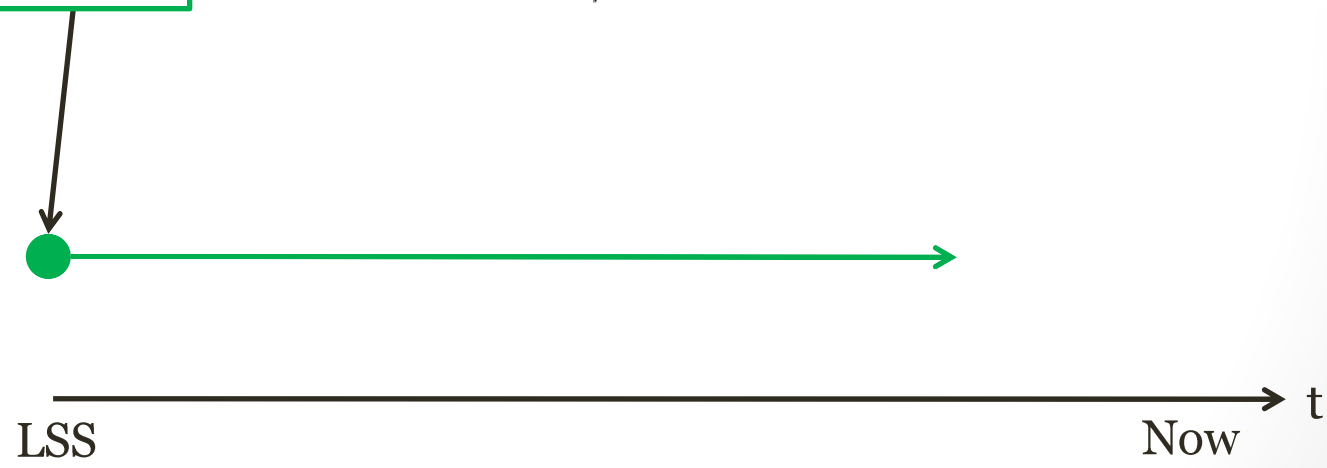

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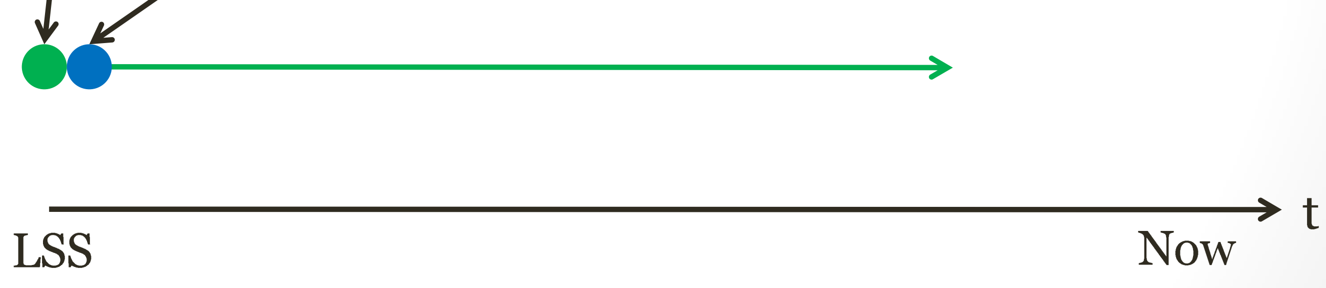

Photon-Redshift Coupling

$$\mathcal{S}_T^{[II]} = -2\left(\frac{\mathrm{d}p^0}{\mathrm{d}\eta}\right)^{[I]}\frac{\partial \mathcal{I}^{[I]}}{\partial p^0} = -2\,p^0\left(\partial_\eta \Psi^{[I]} - n^i\partial_I\Phi^{[I]}\right)\frac{\partial \mathcal{I}^{[I]}}{\partial p^0}$$

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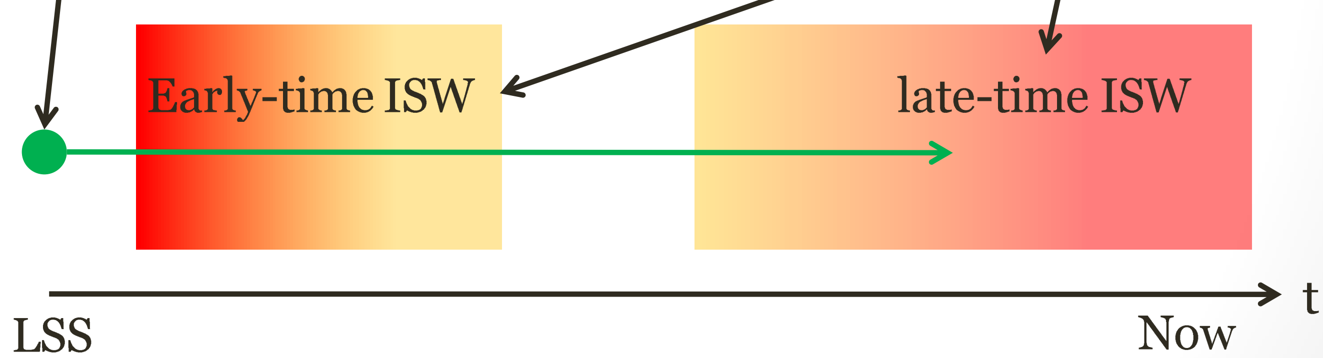
$$\hat{\mathcal{I}}^{[II]} \sim \int_0^{\eta_0} \mathrm{d}\tilde\eta\, e^{-i\mathbf{k}_1\cdot\hat{\mathbf{n}}\tilde r}\,\mathcal{S}_T^{[I]}(\tilde\eta,\mathbf{k}_1,\hat{\mathbf{n}})\left\{e^{-i\mathbf{k}_2\cdot\hat{\mathbf{n}}\tilde r}\Phi(k_2,\tilde\eta) + \int_{\tilde\eta}^{\eta_0} \mathrm{d}\eta_1\, e^{-i\mathbf{k}_2\cdot\hat{\mathbf{n}} r_1}\left[\Phi'(k_2,\eta_1)+\Psi'(k_2,\eta_1)\right]\right\}$$



Photon-Redshift Coupling

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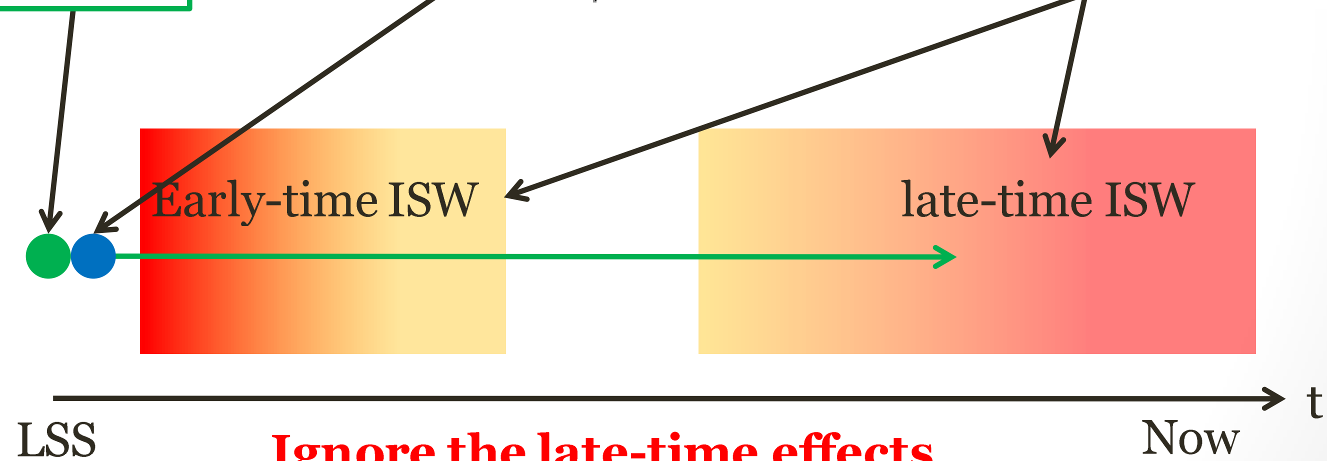

Ignore the late-time effects

- Identify Photon-SW and Photon-ISW couplings
- Separate early-time and late-time effects
- Converge at l~10

Bispectrum around Recombination

No lensing and time-delay

$$\mathcal{S}_T^{[\mathrm{II}]}(\eta, \mathbf{x}, \hat{\mathbf{n}}, p^0) = \mathfrak{C}^{[\mathrm{II}]} + \bar{\tau}' \mathcal{I}^{[\mathrm{II}]} - 2\left(\frac{\mathrm{d}p^0}{\mathrm{d}\eta}\right)^{[\mathrm{I}]} \frac{\partial \mathcal{I}^{[\mathrm{I}]}}{\partial p^0} - \left(\frac{\mathrm{d}p^0}{\mathrm{d}\eta}\right)^{[\mathrm{II}]} \frac{\partial \bar{\mathcal{I}}}{\partial p^0} - 2\left(\frac{\mathrm{d}x^I}{\mathrm{d}\eta}\right)^{[\mathrm{I}]} \frac{\partial \mathcal{I}^{[\mathrm{I}]}}{\partial x^I} - 2\left(\frac{\mathrm{d}n^i}{\mathrm{d}\eta}\right)^{[\mathrm{I}]} \frac{\partial \mathcal{I}^{[\mathrm{I}]}}{\partial n^i}$$

Compton scattering

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2nd-order redshifts

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- Flat-sky & thin-shell

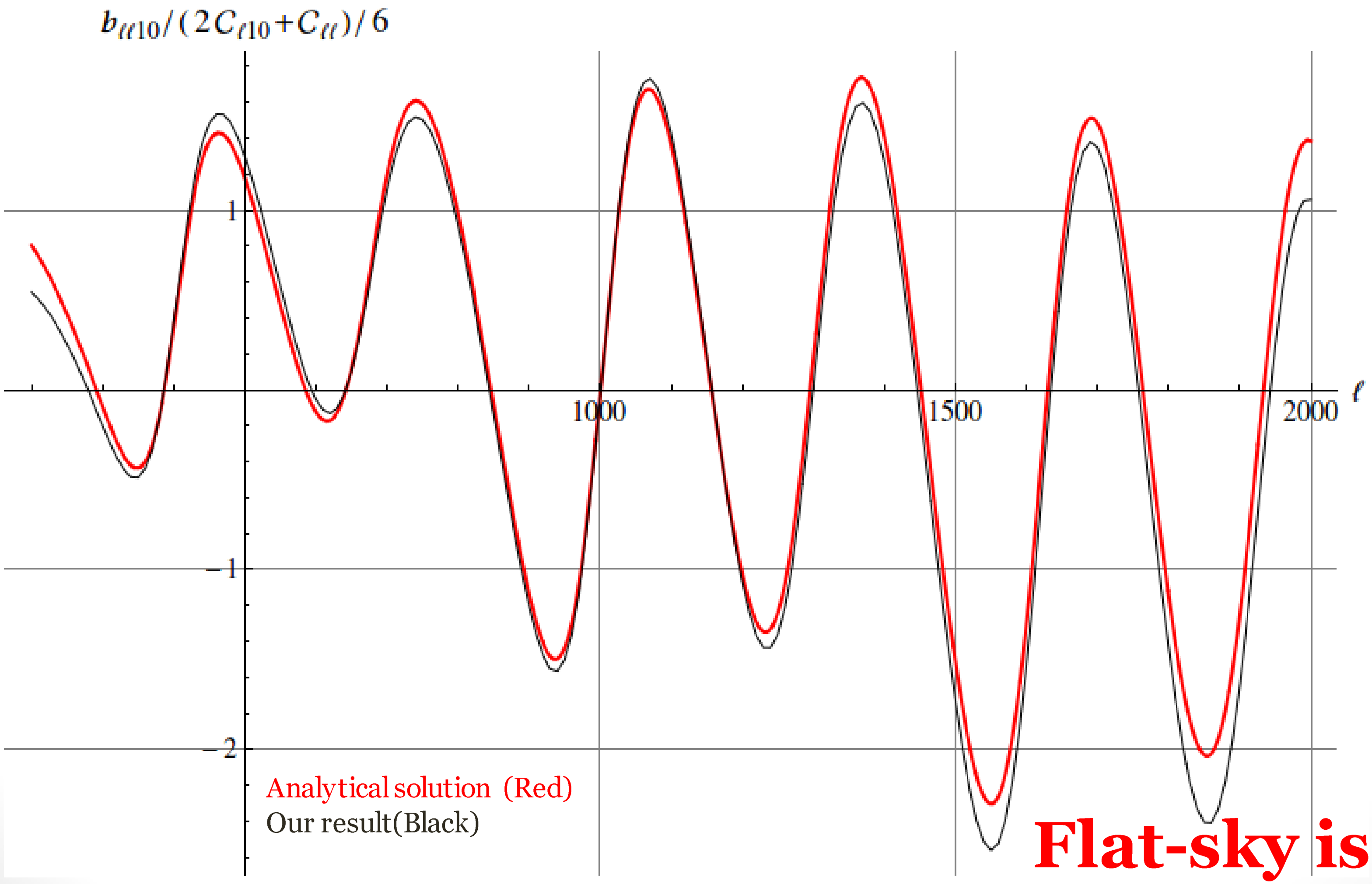
$$b_{l_1 l_2 l_3} \approx \frac{r_{\mathrm{LSS}}^{-4}}{(2\pi)^2} \int_{-\infty}^{\infty} dk_1^z dk_2^z P(k_1) P(k_2) \int_{r_{\mathrm{LSS}}}^{0} dr_1 dr_2 dr_3 e^{-i(k_1^z r_1 + k_2^z r_2 + k_3^z r_3)}$$
$$\mathcal{S}_T^{[\mathrm{I}]}(k_1, r_1) \mathcal{S}_T^{[\mathrm{I}]}(k_2, r_2) \mathcal{S}_T^{[\mathrm{II}]}(\mathbf{k}_1, \mathbf{k}_2, \hat{\mathbf{n}}, r_3) + 1 \leftrightarrow 3 + 2 \leftrightarrow 3$$

$$\mathcal{S}_T^{[\mathrm{II}]}(\mathbf{k}_1, \mathbf{k}_2, \hat{\mathbf{n}}, r) = \sum_{lm} \mathcal{S}_{lm}^{[\mathrm{II}]}(\mathbf{k}_1, \mathbf{k}_2, r) Y_{\hat{\mathbf{k}}}^{lm}(\hat{\mathbf{n}})$$

Include m ≠ 0 modes
Need to be considered outside squeezed limit

Analytical Solution in Squeezed Limit

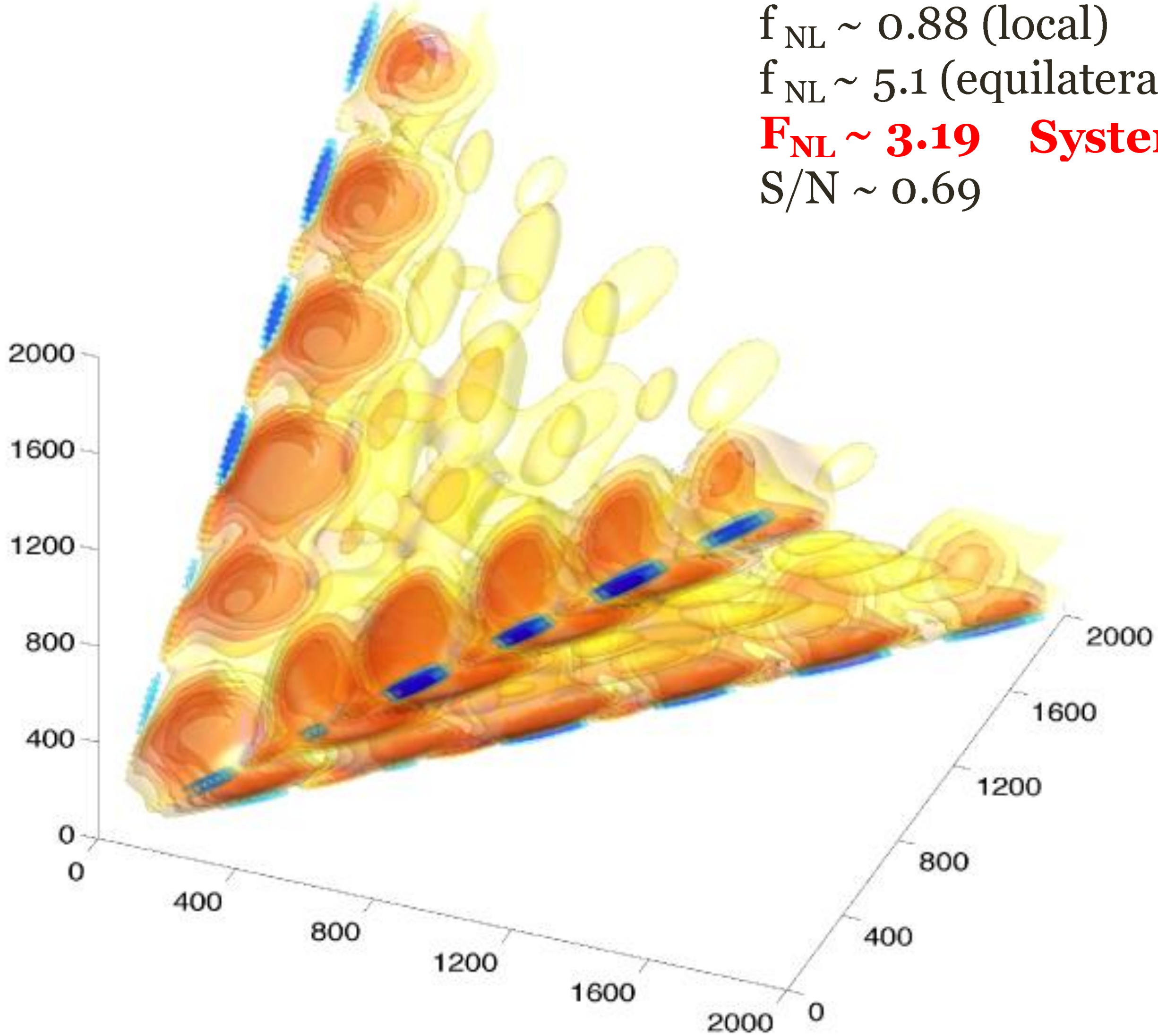
$$b_{l_S l_L l_L} = 2C_{l_S}C_{l_L} + C_{l_L}C_{l_L} - C_{l_S}^{T\zeta}C_{l_L}\frac{d\ln(l_L^2 C_{l_L})}{d\ln l_L} \quad \text{for } l_S = 10$$



Flat-sky is good!

Bispectrum around Recombination

$f_{NL} \sim 0.88$ (local)
$f_{NL} \sim 5.1$ (equilateral)
$F_{NL} \sim 3.19$ Systematic bias to Panck!
S/N ~ 0.69



S.-C. Su, E. A. Lim, and E. P. S. Shellard (2012)

On-going Work

- 2^{nd}-order vector and tensor perturbations

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- Include lensing and time-delay effects

$$\left(\frac{\mathrm{d}x^I}{\mathrm{d}\eta}\right)\frac{\partial \mathscr{P}_{cd}}{\partial x^I}+\left(\frac{\mathrm{d}n^i}{\mathrm{d}\eta}\right)\frac{\partial \mathscr{P}_{cd}}{\partial n^i} \overset{?}{\longleftrightarrow} \tilde{\Theta}(\hat{\mathbf{n}}) = \Theta(\hat{\mathbf{n}}+\boldsymbol{\alpha})$$

  - Correction to 2^{nd}-order lensing from higher-orders ~ 10%

2^{nd}-order Boltzmann equation is not enough!

Formalism has been established!!

First ever attempt!

Lensing in Boltzmann Equations (publish soon)

- Apply line of sight approach iteratively
  - Dyson series

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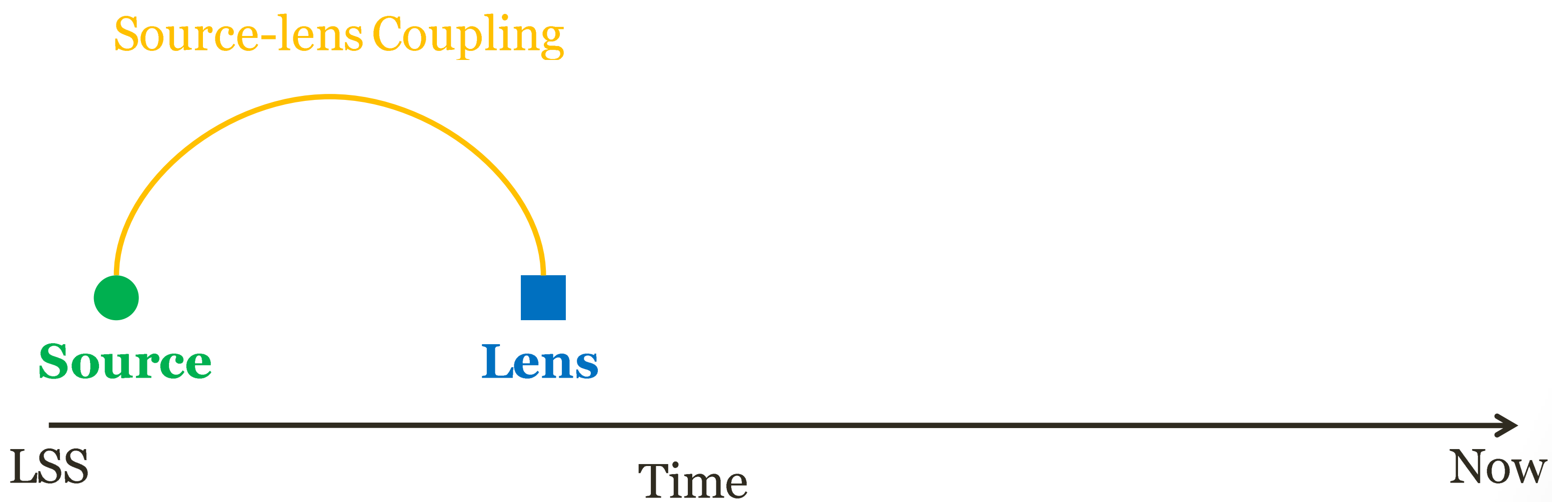

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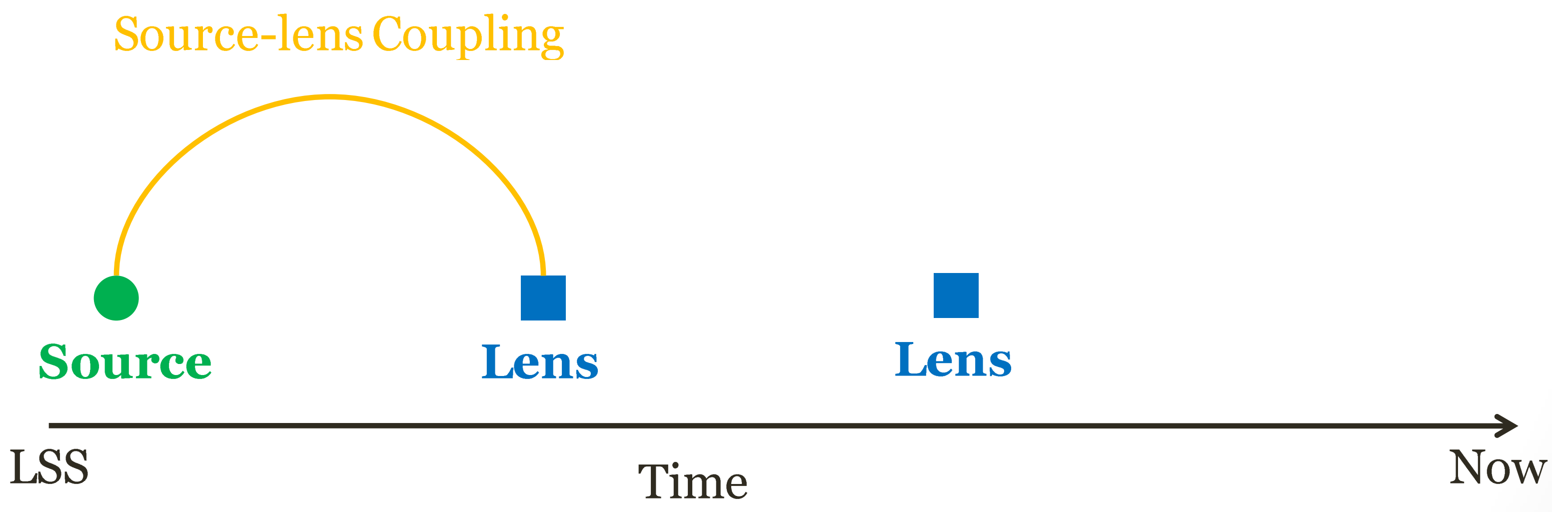

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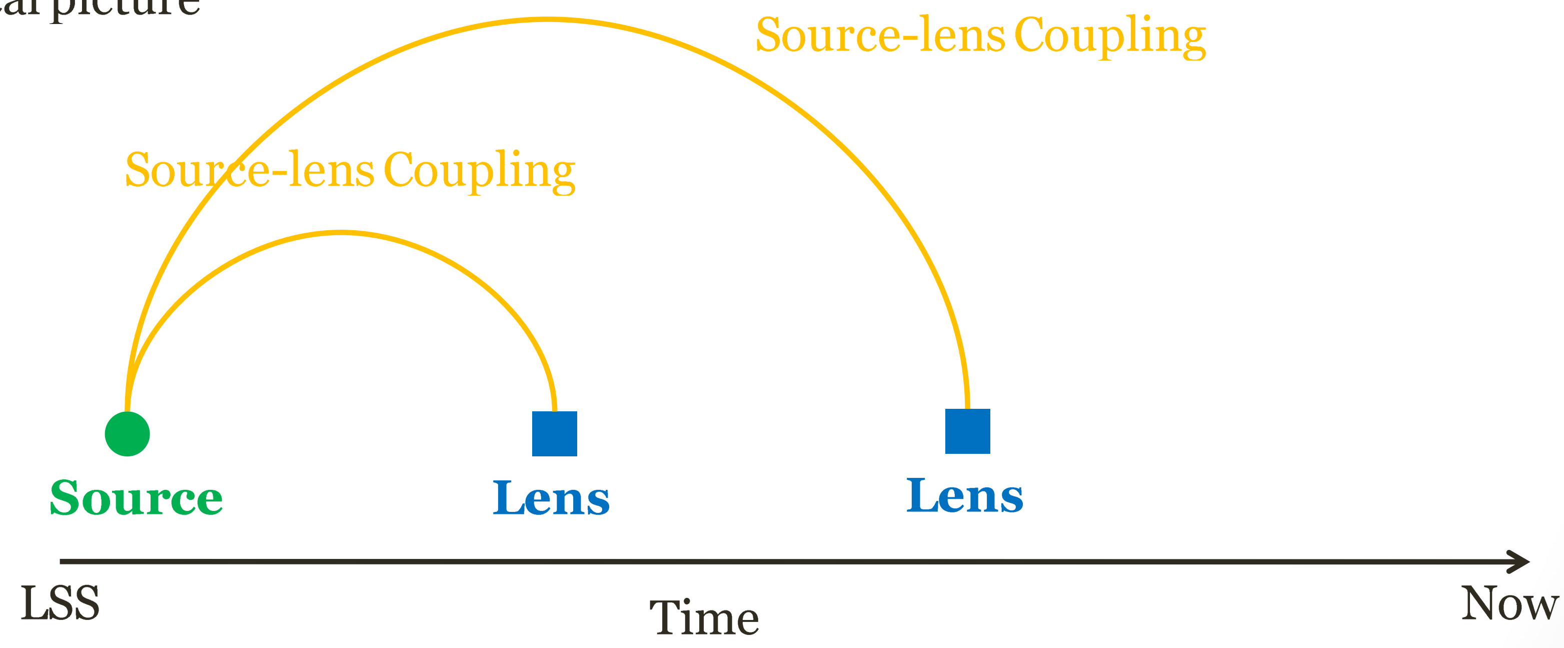

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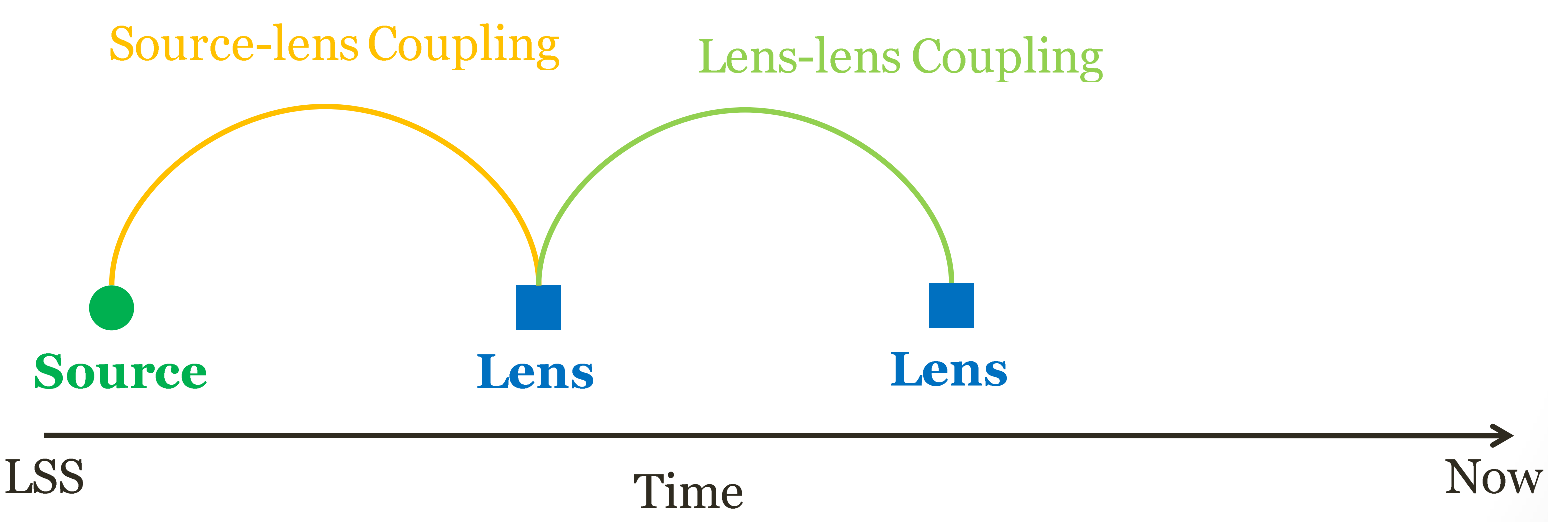

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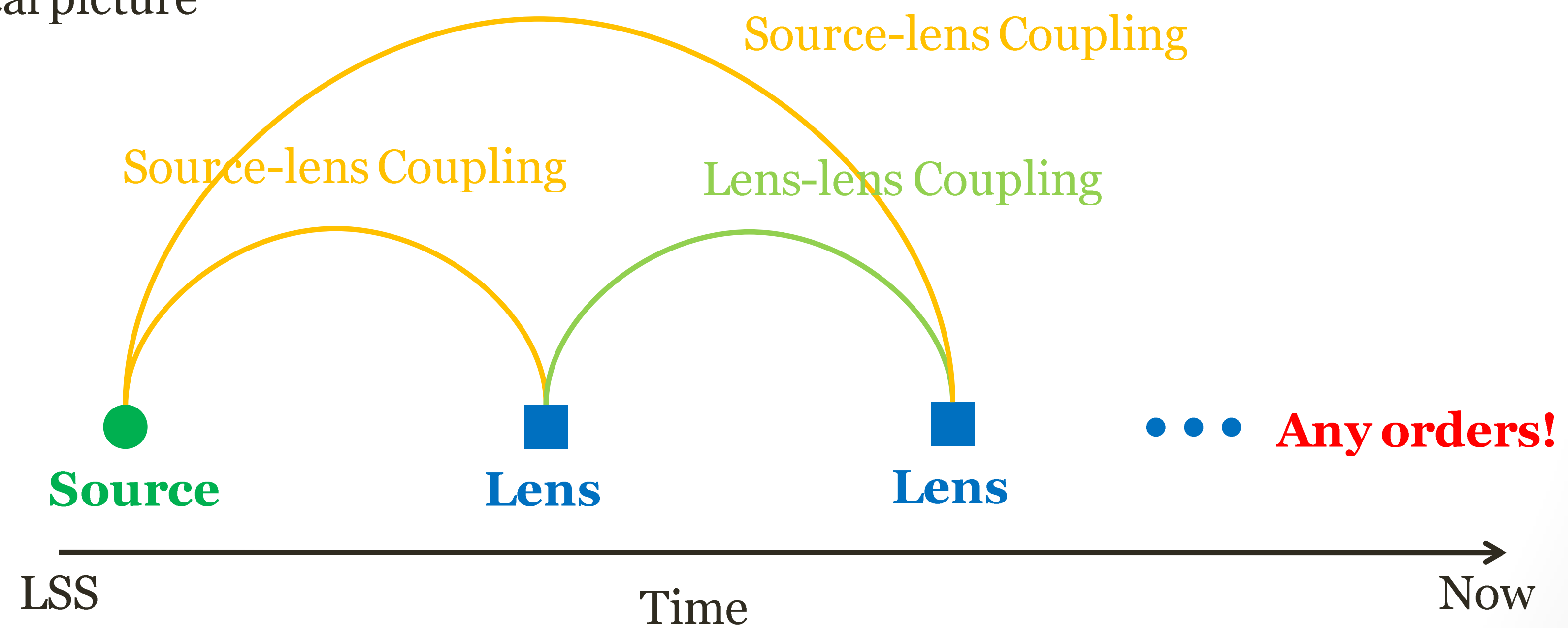

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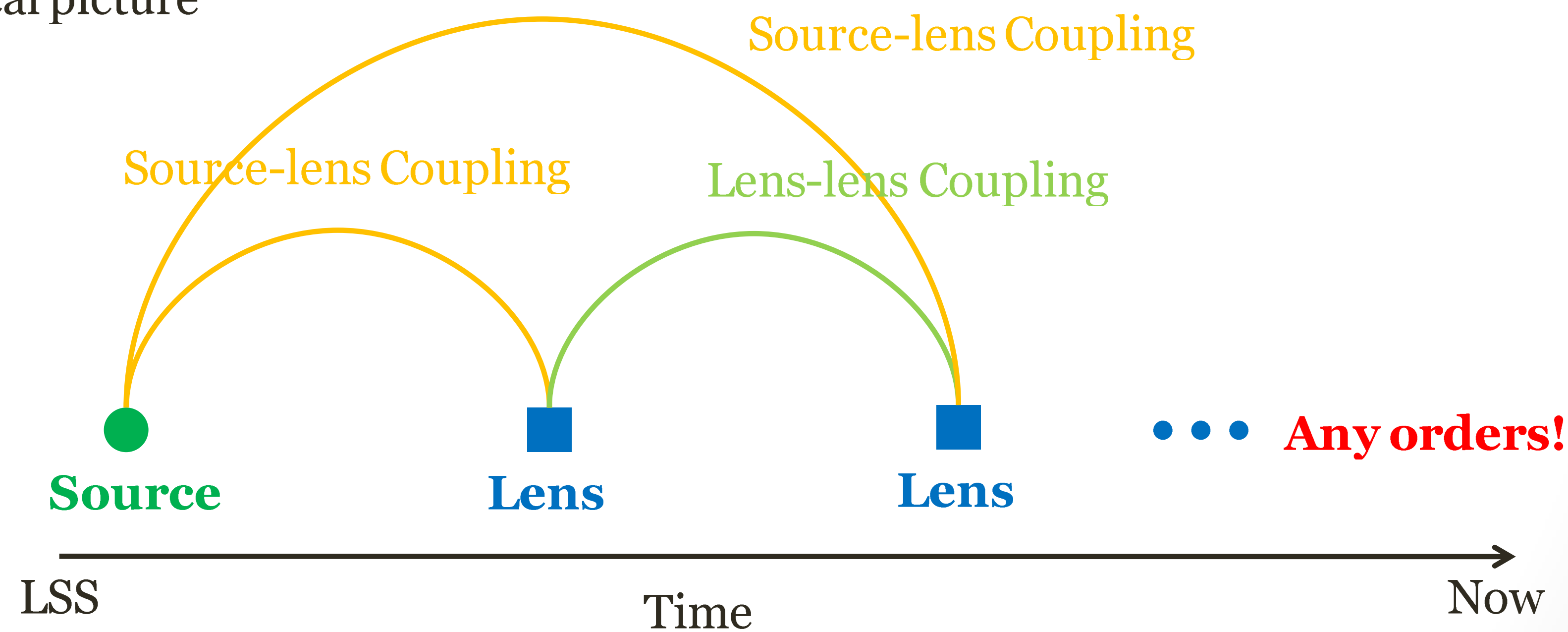

- Extend to time-delay and redshifts
  - Capture all possible couplings systematically!

Are these couplings important!?

Conclusions

- **DONE:**
  - Bispectrum around recombination
  - $\left(\frac{\mathrm{d}x^I}{\mathrm{d}\eta}\right)\frac{\partial \mathscr{P}_{cd}}{\partial x^I}+\left(\frac{\mathrm{d}n^i}{\mathrm{d}\eta}\right)\frac{\partial \mathscr{P}_{cd}}{\partial n^i} \longleftrightarrow \tilde{\Theta}(\hat{\mathbf{n}}) = \Theta(\hat{\mathbf{n}}+\boldsymbol{\alpha})$ Linked!
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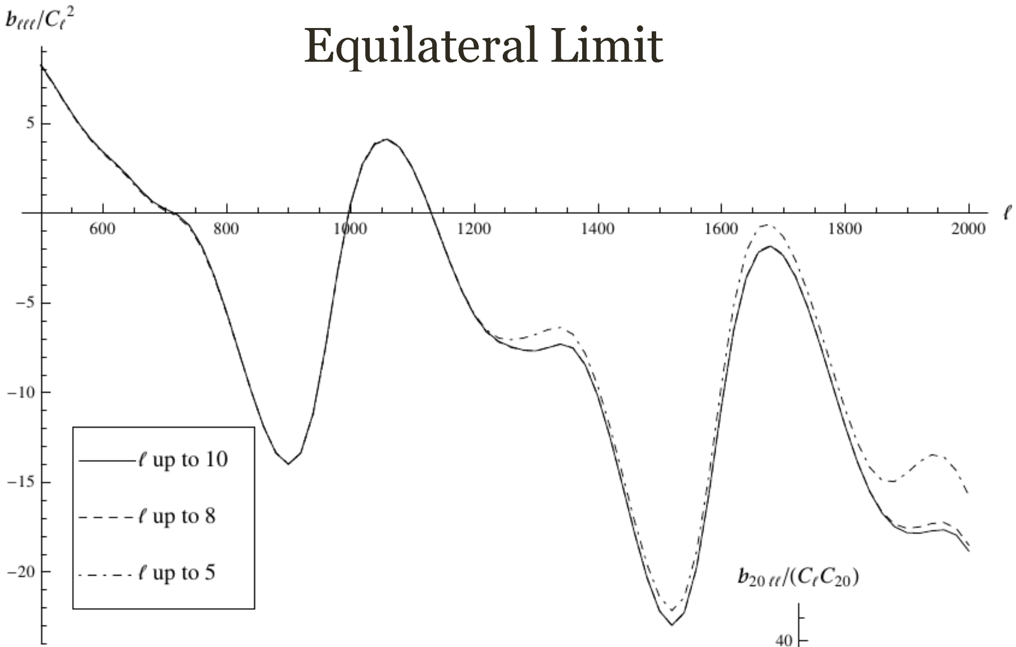
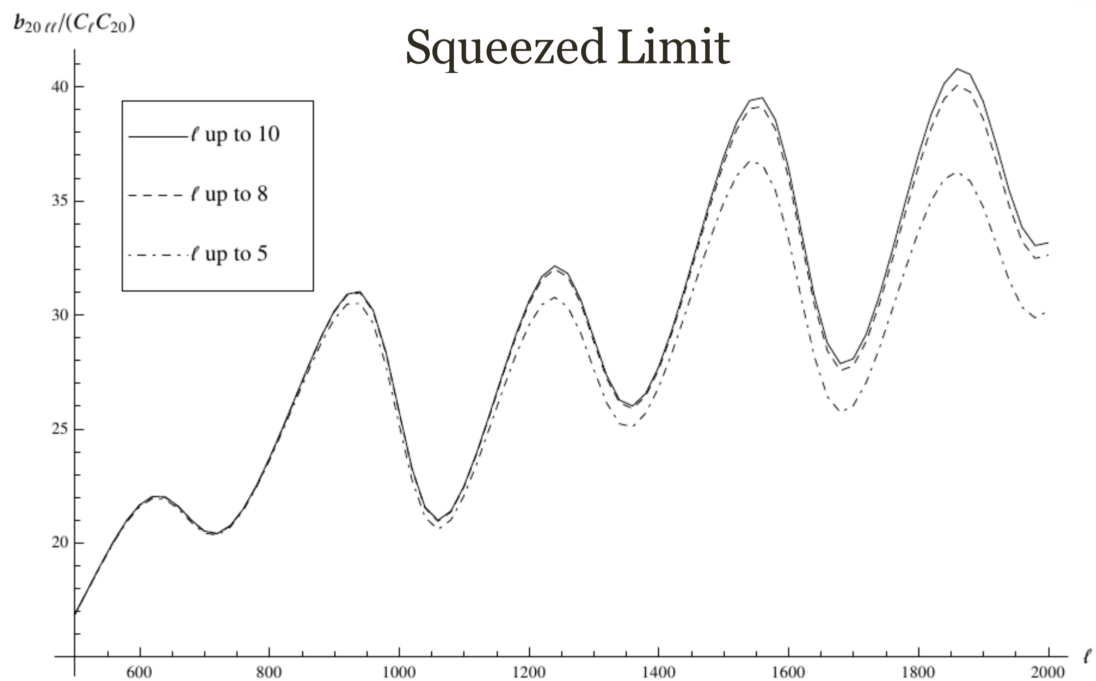
- non-linear effects in Planck: **ISW-lensing bispectrum, high l regimes**
  - PRISM will do even better!

We have entered the "non-linear" CMB era!

Thank You!

Cross-term Source Function





Squeezed Bispectrum