Influence of backreaction on the expansion rate inside LRS class II family

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Outline

Motivation: Cosmology

- Traditional approach

$$E_{\mu\nu}\left(\langle g_{\alpha\beta}\rangle\right) = 8\pi\left\langle T_{\mu\nu}\right\rangle$$

$$\left\langle T_{\mu\nu}\right\rangle = \rho u_\mu u_\nu + p\left(g_{\mu\nu} + u_\mu u_\nu\right)$$

$$\left\langle g_{\mu\nu}\right\rangle = -dt^2 + a^2\left(t\right)\left[d\chi^2 + \Sigma^2 d\Omega^2\right]$$

- ”Correct” approach

$$\left\langle E_{\mu\nu}\left(g_{\alpha\beta}\right)\right\rangle = 8\pi\left\langle T_{\mu\nu}\right\rangle$$

$$E_{\mu\nu}(\langle g_{\mu\nu}\rangle) = 8\pi\left\langle T_{\mu\nu}\right\rangle + E_{\mu\nu}(\langle g_{\mu\nu}\rangle) - \langle E_{\mu\nu}(g_{\mu\nu})\rangle$$

- How to average tensor?

Buchert equations

- For the metric $ds^2 = -dt^2 + g_{ij}dX^i dX^j$ spatial averaging of the scalar field Ψ over the domain $\mathcal{D}$ is defined by

$$\langle \Psi(t, X^i) \rangle_{\mathcal{D}} := \frac{1}{V_{\mathcal{D}}} \int_{\mathcal{D}} \sqrt{\det g_{ij}} d^3 X \Psi(t, X^i)$$

- To obtain scalar equation from the Einstein equation, we have to contract it with available tensors - $g^{\mu\nu}$, u^μ and ∇^μ

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} + 4\pi G \langle \rho \rangle_{\mathcal{D}} - \Lambda = \mathcal{Q}_{\mathcal{D}}$$

$$\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right)^2 - \frac{8\pi G}{3} \langle \rho \rangle_{\mathcal{D}} + \frac{\langle \mathcal{R} \rangle_{\mathcal{D}}}{6} - \frac{\Lambda}{3} = -\frac{\mathcal{Q}_{\mathcal{D}}}{6}$$

$$\partial_t \langle \rho \rangle_{\mathcal{D}} + 3\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \langle \rho \rangle_{\mathcal{D}} = 0$$

with the backreaction term

$$\mathcal{Q}_{\mathcal{D}} := \frac{2}{3}\left(\left\langle \Theta^2 \right\rangle_{\mathcal{D}} - \langle \Theta \rangle_{\mathcal{D}}^2\right) - 2\left\langle \sigma^2 \right\rangle_{\mathcal{D}}$$

Buchert equations

- dimensionless scale factor $a_{\mathcal{D}}$ and the effective Hubble parameter $H_{\mathcal{D}}$ are defined as

$$a_{\mathcal{D}} = \left(\frac{V_{\mathcal{D}}}{V_{\mathcal{D}i}}\right)^{\frac{1}{3}}$$

$$\langle\Theta\rangle_{\mathcal{D}} = \frac{\dot{V}_{\mathcal{D}}}{V_{\mathcal{D}}} = 3\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} =: 3H_{\mathcal{D}}$$

- In a similar way as in the FRW approach we can define dimensionless variables (omega factors)

$$\Omega_m^{\mathcal{D}} := \frac{8\pi G}{3H_{\mathcal{D}}^2}\langle\rho\rangle_{\mathcal{D}}\,; \quad \Omega_\Lambda^{\mathcal{D}} := \frac{\Lambda}{3H_{\mathcal{D}}^2}; \quad \Omega_{\mathcal{R}}^{\mathcal{D}} := -\frac{\langle\mathcal{R}\rangle_{\mathcal{D}}}{6H_{\mathcal{D}}^2}; \quad \Omega_{\mathcal{Q}}^{\mathcal{D}} := -\frac{\mathcal{Q}_{\mathcal{D}}}{6H_{\mathcal{D}}^2}$$

and Hamiltonian constraint will be written in the standard form

$$\Omega_m^{\mathcal{D}} + \Omega_\Lambda^{\mathcal{D}} + \Omega_{\mathcal{R}}^{\mathcal{D}} + \Omega_{\mathcal{Q}}^{\mathcal{D}} = 1$$

LRS spacetime

- van Elst, Ellis, *Class. Quantum Grav.* 13 1099 (1996).
- Definition: In an open neighborhood of the each point p, there exists a nondiscrete subgroup of the Lorentz group which leaves the Riemann tensor and its covariant derivatives invariant.
- In LRS spacetimes therefore exists the preferred direction e^{μ} (the axis of symmetry) in every point.

$$e_{\rho}u^{\rho} = 0, \quad e_{\rho}e^{\rho} = 1.$$

LRS spacetime

- Because of property of the LRS spacetime, all covariantly defined spacelike vectors orthogonal to u^μ must be proportional to e^μ

$$\dot{u}^\mu = \dot{u} e^\mu, \qquad \omega^\mu = \omega e^\mu.$$

$$h^\sigma_{\ \mu} \nabla_\sigma \rho = \rho' e_\mu, \quad h^\sigma_{\ \mu} \nabla_\sigma p = p' e_\mu, \quad h^\sigma_{\ \mu} \nabla_\sigma \theta = \theta' e_\mu.$$

- We can define the magnitude of the spatial rotation k and the magnitude of the spatial divergence a:

$$k := \left| \eta^{\alpha\beta\gamma\delta} \left(\nabla_\beta e_\gamma \right) u_\delta \right|$$

$$a := h^\alpha_\beta \left(\nabla_\alpha e^\beta \right)$$

- We can create a new tensor $e_{\mu\nu}$ defined from e_μ

$$e_{\mu\nu} := \frac{1}{2} \left(3 e_\mu e_\nu - h_{\mu\nu} \right)$$

$$\sigma_{\mu\nu} = \frac{2}{\sqrt{3}} \sigma e_{\mu\nu}, \qquad E_{\mu\nu} = \frac{2}{\sqrt{3}} E e_{\mu\nu}, \qquad H_{\mu\nu} = \frac{2}{\sqrt{3}} H e_{\mu\nu}.$$

Dust LRS class II

- $k = \omega = 0 \Rightarrow H = 0$

$$\dot{\theta} = -\frac{1}{3}\theta^2 - 2\sigma^2 - 4\pi\rho,$$

$$\dot{\sigma} = -\frac{1}{\sqrt{3}}\sigma^2 - \frac{2}{3}\theta\sigma - E,$$

$$\dot{E} = -4\pi\rho\sigma + \sqrt{3}E\sigma - \theta E,$$

$$\dot{\rho} = -\rho\theta,$$

$$\dot{a} = -\frac{1}{3}a\theta + \frac{1}{\sqrt{3}}a\sigma.$$

$$\sigma' = \frac{1}{\sqrt{3}}\theta' - \frac{2}{3}a\sigma,$$

$$E' = -\frac{3}{2}aE + \frac{4\pi}{\sqrt{3}}\rho',$$

$$a' = \frac{2}{9}\theta^2 + \frac{2}{3\sqrt{3}}\theta\sigma - \frac{4}{3}\sigma^2 - \frac{2}{\sqrt{3}}E - \frac{1}{2}a^2 - \frac{16\pi}{3}\rho.$$

Averaging dust LRS class II

- We will use averaging over the spacelike domain $\mathcal{D}$

$$\langle A\rangle_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}}}\int\limits_{\mathcal{D}} d^3X\sqrt{\det g_{ij}}A$$

- Averaging evolution equations:

$$\langle\theta\rangle^{\cdot} = -\frac{1}{3}\langle\theta\rangle^2 - 4\pi\langle\rho\rangle + \underline{\frac{2}{3}\left(\left\langle\theta^2\right\rangle - \langle\theta\rangle^2\right) - 2\left\langle\sigma^2\right\rangle}$$

$$\langle\sigma\rangle^{\cdot} = -\frac{1}{\sqrt{3}}\langle\sigma\rangle^2 - \frac{2}{3}\langle\theta\rangle\langle\sigma\rangle - \langle E\rangle + \underline{\frac{1}{\sqrt{3}}\left(\langle\sigma\rangle^2 - \left\langle\sigma^2\right\rangle\right)}$$
$$\underline{+\frac{1}{3}\left(\langle\theta\sigma\rangle - \langle\theta\rangle\langle\sigma\rangle\right)}$$

$$\langle E\rangle^{\cdot} = -4\pi\langle\rho\rangle\langle\sigma\rangle + \sqrt{3}\langle E\rangle\langle\sigma\rangle - \langle\theta\rangle\langle E\rangle$$
$$\underline{-4\pi\left(\langle\rho\sigma\rangle - \langle\rho\rangle\langle\sigma\rangle\right) + \sqrt{3}\left(\langle E\sigma\rangle - \langle E\rangle\langle\sigma\rangle\right)}$$

$$\langle\rho\rangle^{\cdot} = -\langle\rho\rangle\langle\theta\rangle$$

Averaging dust LRS class II

$$\langle a\rangle^{\cdot} = -\frac{1}{3}\langle a\rangle\langle\theta\rangle + \frac{1}{\sqrt{3}}\langle a\rangle\langle\sigma\rangle + \underline{\frac{2}{3}\left(\langle a\theta\rangle - \langle a\rangle\langle\theta\rangle\right)}$$

$$\underline{+\frac{1}{\sqrt{3}}\left(\langle a\sigma\rangle - \langle a\rangle\langle\sigma\rangle\right)}$$

and constrains

$$\langle\sigma\rangle' = \frac{1}{\sqrt{3}}\langle\theta\rangle' - \frac{2}{3}\langle a\rangle\langle\sigma\rangle + \underline{\langle\sigma\xi\rangle - \langle\xi\rangle\langle\sigma\rangle - \frac{1}{\sqrt{3}}\left(\langle\xi\theta\rangle - \langle\xi\rangle\langle\theta\rangle\right)}$$

$$\underline{-\frac{3}{2}\left(\langle a\sigma\rangle - \langle a\rangle\langle\sigma\rangle\right)}$$

$$\langle E\rangle' = -\frac{2}{3}\langle a\rangle\langle E\rangle + \frac{4\pi}{\sqrt{3}}\langle\rho\rangle' - \underline{\frac{2}{3}\left(\langle aE\rangle - \langle a\rangle\langle E\rangle\right)}$$

$$\underline{+\langle\xi E\rangle - \langle\xi\rangle\langle E\rangle - \frac{4\pi}{\sqrt{3}}\left(\langle\xi\rho\rangle - \langle\xi\rangle\langle\rho\rangle\right)}$$

Averaging dust LRS class II

$$\langle a\rangle' = \frac{2}{9}\langle\theta\rangle^2 + \frac{2}{3\sqrt{3}}\langle\theta\rangle\langle\sigma\rangle - \frac{4}{3}\langle\sigma\rangle^2 - \frac{2}{\sqrt{3}}\langle E\rangle - \frac{1}{2}\langle a\rangle^2 - \frac{16\pi}{3}\langle\rho\rangle$$

$$\underline{+\langle a\xi\rangle - \langle a\rangle\langle\xi\rangle + \frac{2}{9}\left(\left\langle\theta^2\right\rangle - \langle\theta\rangle^2\right) + \frac{2}{3\sqrt{3}}\left(\langle\theta\sigma\rangle - \langle\theta\rangle\langle\sigma\rangle\right)}$$

$$\underline{-\frac{4}{3}\left(\left\langle\sigma^2\right\rangle - \langle\sigma\rangle^2\right) - \frac{1}{2}\left(\left\langle a^2\right\rangle - \langle a\rangle^2\right)}$$

where we used the commutation relation

$$\langle A\rangle'_{\mathcal{D}} = -\langle\xi\rangle_{\mathcal{D}}\langle A\rangle_{\mathcal{D}} + \langle A\xi\rangle_{\mathcal{D}} + \left\langle A'\right\rangle_{\mathcal{D}},$$

with the function ξ defined by $\left(\sqrt{\det g_{ij}}\right)' = \sqrt{\det g_{ij}}\xi$.

Effective Friedmann equations

- If you multiply unaveraged equations by 2θ (or 2ω) you can obtain evolution equations for θ^2 (or ω^2)
- Choose one representative point and perform averaging over some region Ω. You obtain only time dependent quantities.
- The same procedure could be done for higher order terms. You obtain infinite set of equations, which you can truncate at given power (higher order correlation terms are usually much smaller) or you can assume some relations among averaged quantities.
- e.g evolution equation for θ^2 reads

$$\left\langle \theta^2 \right\rangle^{\cdot}_{\Omega} = -\frac{2}{3}\left\langle \theta \right\rangle^3_{\Omega} - 4\left\langle \theta \right\rangle_{\Omega}\left\langle \sigma \right\rangle^2_{\Omega} - 8\pi\left\langle \rho \right\rangle_{\Omega}\left\langle \theta \right\rangle_{\Omega} + \underline{\frac{1}{3}\left(\left\langle \theta^3 \right\rangle_{\Omega} - \left\langle \theta \right\rangle^3_{\Omega}\right)}$$

$$\underline{+\left(\left\langle \theta \right\rangle^3_{\Omega} - \left\langle \theta \right\rangle_{\Omega}\left\langle \theta^2 \right\rangle_{\Omega}\right) - 4\left(\left\langle \theta\sigma^2 \right\rangle_{\Omega} - \left\langle \theta \right\rangle_{\Omega}\left\langle \sigma \right\rangle^2_{\Omega}\right)}$$

$$\underline{-4\pi\left(\left\langle \rho\theta \right\rangle_{\Omega} - \left\langle \rho \right\rangle_{\Omega}\left\langle \theta \right\rangle_{\Omega}\right)}$$

Conclusion

- We have presented the averaged equations for dust LRS class II family
- Evolution equation for expansion scalar looks like Friedmann equation with additional terms.
- Inhomogeneity are modeled by the ansatz of the correlation function - some of them could be neglected or proportional to each other.

Thank You!