On the non-linear scale of cosmological perturbation theory

Mathias Garny (DESY Hamburg / CERN)

Cosmo Cambridge, 02.09.13

based on 1304.1546 and 1309.xxxx
with Diego Blas, Thomas Konstandin
Motivation

- Matter power spectrum in the regime of baryon acoustic oscillations will be measured with high precision (Euclid, . . .)
- Desirable to develop a (fast) method for theoretical prediction for a given set of parameters (+ analytic understanding of onset of non-linearities)
- Weakly non-linear regime ⇒ borderline for perturbation theory

\[ \Delta^2(k, z) = 4\pi k^3 P(k, z) \]

Kuhlen, Vogelsberger, Angulo 1209.5745
Outline

- Standard Perturbation Theory
- Three loop result
- Padé resummation
- Padé improved PT
Standard Perturbation Theory

- Poisson/Euler/Continuity eq. for density contrast \( \delta = \rho/\bar{\rho} - 1 \) and pec. velocity (neglect vorticity/viscosity \( \rightarrow \) talks by Mercolli, Zaldarriaga)

- Expand solution in \( \delta_0(k) = \delta(k, z_0 \sim 10^3) \), for EdS / growing mode

\[
\delta(k, z) = \sum_{n=1}^{\infty} (D_+(z))^n \int \delta^{(3)}(k - \sum q_i) F_n(q_1, \ldots, q_n) \delta_0(q_1) \cdots \delta_0(q_n)
\]

\[
= D_+(z) \delta_0(k) + \ldots
\]

e.g. \( F^s_2(q_1, q_2) = \frac{5}{7} + \frac{1}{2} \frac{q_1 \cdot q_2}{q_1 q_2} \left( \frac{q_1}{q_2} + \frac{q_2}{q_1} \right) + \frac{2}{7} \left( \frac{q_1 \cdot q_2}{q_1 q_2} \right)^2 \)
Standard Perturbation Theory

- Poisson/Euler/Continuity eq. for density contrast \( \delta = \rho/\bar{\rho} - 1 \) and pec. velocity (neglect vorticity/viscosity \( \rightarrow \) talks by Mercolli, Zaldarriaga)

- Expand solution in \( \delta_0(k) = \delta(k, z_0 \sim 10^3) \), for EdS / growing mode

\[
\delta(k, z) = \sum_{n=1}^{\infty} (D_+(z))^n \int \delta^{(3)}(k - \sum q_i) F_n(q_1, \ldots, q_n) \delta_0(q_1) \cdots \delta_0(q_n)
\]

\[
= D_+(z) \delta_0(k) + \ldots
\]

- Power spectrum \( \langle \delta(k, z) \delta(k', z) \rangle = \delta^{(3)}(k + k') P(k, z) \) is obtained using Wick theorem in terms of \( P_0(q) = P(q, z_0) \) for Gaussian IC

\[
P(k, z) = \underbrace{P_{\text{lin}}}_{D_+(z)^2 P_0(k)} + \underbrace{P_{1-\text{loop}}}_{D_+(z)^4 (2P_{13} + P_{22}) + D_+(z)^6 (2P_{15} + 2P_{24} + P_{33}) + \ldots}
\]

- e.g. \( P_{22}(k) = 2 \int d^3 q \ F_2^s(q, k - q)^2 \ P_0(q) P_0(|k - q|) \)

Mathias Garny (DESY Hamburg / CERN)  On the non-linear scale of cosmological perturbation theory
SPT breaks down at small scales / late times \[ P_{L-loop} \propto D_+^{2L+2} \sim \left( \frac{1}{1+z} \right)^{2L+2} \] initial spectrum from CAMB (ΛCDM - WMAP5)
• Many approaches have been developed to improve behaviour at large $k$ (closure, LPT, RPT, eikonal, ...) see e.g. Carlson, White, Padmanabhan 0905.0479

• Enhanced contributions ($\propto k^{2n}, n \leq L$) related to interaction with soft modes cancel out completely in equal-time correlators (like the PS), as expected from Galilean invariance

  see e.g. Bertschinger, Jain 95; Frieman, Scoccimarro 95; Anselmi, Pietroni 12; Pietroni, Peloso 13; Sugiyama, Futamase 13; Blas, MG, Konstandin 13; Carrasco, Foreman, Green, Senatore 13; …
Many approaches have been developed to improve behaviour at large $k$ (closure, LPT, RPT, eikonal, ...)

see e.g. Carlson, White, Padmanabhan 0905.0479

Enhanced contributions ($\propto k^{2n}$, $n \leq L$) related to interaction with soft modes cancel out completely in equal-time correlators (like the PS), as expected from Galilean invariance

see e.g. Bertschinger, Jain 95; Frieman, Scoccimarro 95; Anselmi, Pietroni 12; Pietroni, Peloso 13; Sugiyama, Futamase 13; Blas, MG, Konstandin 13; Carrasco, Foreman, Green, Senatore 13; ...

$\Rightarrow$ RPT ‘as good as’ SPT for large-$k$ for equal-time correlators (but very useful for unequal-time corr. as e.g. propagator)

ea.g. Sugiyama, Spergel 13

Cancellations (leading + sub-leading terms) can be made manifest for loop integrand

$\Rightarrow$ crucial for Monte-Carlo integration
Three loop

Diego Blas, MG, Thomas Konstandin 1309.xxxx (power spectrum); see also Bernardeau, Taruya, Nishimichi, 1211.1571 (propagator)

Mathias Garny (DESY Hamburg / CERN)  On the non-linear scale of cosmological perturbation theory
Loop expansion of PS in the limit of small $k$

For small $k$

\[
P_{L\text{- loop}}(k, z) \rightarrow \frac{(2L + 1)!}{2^{L-1} L!} P_{\text{lin}}(k, z) \int_{q_1} \cdots \int_{q_L} F_{2L+1}^s(k, q_1, -q_1, \ldots, q_L, -q_L) \times P_{\text{lin}}(q_1, z) \cdots P_{\text{lin}}(q_L, z)
\]

Using property $F_{2L+1}^s \propto k^2$ for $k \ll q_i$

\[
P_{L\text{- loop}}(k, z) \propto k^2 P_{\text{lin}}(k, z)
\]

for small $k$

---

Mathias Garny (DESY Hamburg / CERN)  
On the non-linear scale of cosmological perturbation theory
One, two and three loop normalized to $k^2 P_{\text{lin}}(k, z)$

Mathias Garny (DESY Hamburg / CERN)  
On the non-linear scale of cosmological perturbation theory

\[ P_{L-\text{loop}}(k, z) \propto k^2 P_{\text{lin}}(k, z) \]

up to $k = 0.003, 0.06, 0.08 \ h/\text{Mpc}$ for 1, 2, 3-loop at %-level
Loop expansion of PS in the limit of small $k$

Using property $F_{2L+1}^s \propto k^2/q^2$ for $k \ll q_i$ and $q = \max(q_i)$

$$P_{L-\text{loop}} \rightarrow P_{L-\text{small-k}} = -\frac{244\pi}{325} k^2 P_{\text{lin}}(k, z) \times C_L \times \int_0^\infty dq P_{\text{lin}}(q, z) \sigma_i^{2L-2}(q, z)$$

with coeff. $C_L$ ($C_1 = 1$, $C_2 \simeq 0.71$, $C_3 \simeq 1.05$) and scale-dep. variance

$$\sigma_i^2(q, z) \equiv 4\pi \int_0^q dp \, p^2 P_{\text{lin}}(p, z)$$
Loop expansion of PS in the limit of small $k$

Using property $F_{s}^{2L+1} \propto k^2/q^2$ for $k \ll q_i$ and $q = \max(q_i)$

$$P_{L-\text{loop}} \rightarrow P_{L-\text{loop}}^{\text{small}-k} = -\frac{244\pi}{325} k^2 P_{\text{lin}}(k, z) \times C_L \times \int_0^{\infty} dq P_{\text{lin}}(q, z) \sigma_i^{2L-2}(q, z)$$

with coeff. $C_L$ ($C_1 = 1$, $C_2 \approx 0.71$, $C_3 \approx 1.05$) and scale-dep. variance

$$\sigma_i^2(q, z) \equiv 4\pi \int_0^q dp \, p^2 P_{\text{lin}}(p, z)$$

Estimate for Eisenstein-Hu spectrum with $n_s \approx 1$

$$P_{L-\text{loop}}^{\text{small}-k} \propto k^2 P_{\text{lin}}(k, z) \times C_L \times \frac{(3L-1)!}{2^{3L}} D_+(z)^{2L}$$

$\Rightarrow$ Loop expansion is divergent series even at small $k$ and for any $z$
Loop expansion of PS in the limit of small $k$

Using property $F^s_{2L+1} \propto k^2/q^2$ for $k \ll q_i$ and $q = \text{max}(q_i)$

$$P_{L-\text{loop}} \rightarrow P_{L-\text{loop}}^{\text{small} - k} = -\frac{244\pi}{325} k^2 P_{\text{lin}}(k, z) \times C_L \times \int_0^\infty dq P_{\text{lin}}(q, z) \sigma_i^{2L-2}(q, z)$$

with coeff. $C_L$ ($C_1 = 1$, $C_2 \simeq 0.71$, $C_3 \simeq 1.05$) and scale-dep. variance

$$\sigma_i^2(q, z) \equiv 4\pi \int_0^q dp \, p^2 P_{\text{lin}}(p, z)$$

Estimate for Eisenstein-Hu spectrum with $n_s \simeq 1$

$$P_{L-\text{loop}}^{\text{small} - k} \propto k^2 P_{\text{lin}}(k, z) \times C_L \times \frac{(3L - 1)!}{2^{3L}} D_+(z)^{2L}$$

⇒ Loop expansion is divergent series even at small $k$ and for any $z$

* Terms decrease up to a certain order $L_{\text{max}}(z)$, then increase
* Typical behaviour of an asymptotic series (e.g. loop exp. in QED)
Loop expansion of PS in the limit of small $k$

Using property $F_{2L+1}^s \propto k^2/q^2$ for $k \ll q_i$ and $q = \text{max}(q_i)$

$$P_{L-\text{loop}} \to P_{L-\text{loop}}^{\text{small}-k} = -\frac{244\pi}{325} k^2 P_{\text{lin}}(k, z) \times C_L \times \int_0^\infty dq P_{\text{lin}}(q, z) \sigma_i^{2L-2}(q, z)$$

with coeff. $C_L$ ($C_1 = 1$, $C_2 \sim 0.71$, $C_3 \sim 1.05$) and scale-dep. variance

$$\sigma_i^2(q, z) \equiv 4\pi \int_0^q dp p^2 P_{\text{lin}}(p, z)$$

Estimate for Eisenstein-Hu spectrum with $n_s \sim 1$

$$P_{L-\text{loop}}^{\text{small}-k} \propto k^2 P_{\text{lin}}(k, z) \times C_L \times \frac{(3L-1)!}{2^{3L}} D_+(z)^{2L}$$

⇒ Loop expansion is divergent series even at small $k$ and for any $z$

- Terms decrease up to a certain order $L_{\text{max}}(z)$, then increase
- Typical behaviour of an asymptotic series (e.g. loop exp. in QED)
- Partial sum up to smallest term yields best result, with error of order the smallest term (e.g. $P_{2-\text{loop}}/P_{\text{lin}} \sim 6\%$ at $z = 0$, $k = 0.1h/\text{Mpc}$)
Padé ansatz

Goal: improve convergence to go to 90%-accuracy
Idea: resummation in small-$k$ limit

\[ P_{\text{small-}k}(k, z) = -\frac{244\pi}{315} k^2 P_{\text{lin}}(k, z) \times \int_0^\infty dq P_{\text{lin}}(q, z) K(\sigma^2_i(q, z)) \]

where the integrand kernel $K$ is given by a series in $x \equiv \sigma^2_i(q, z)$,

\[ K(x) = \sum_{L=1}^\infty C_L x^{L-1} \]
Padé ansatz

Goal: improve convergence to go to %-accuracy
Idea: resummation in small-$k$ limit

$$P_{\text{small}-k}(k, z) = -\frac{244\pi}{315} k^2 P_{\text{lin}}(k, z) \times \int_0^\infty dq P_{\text{lin}}(q, z) K(\sigma_i^2(q, z))$$

where the integrand kernel $K$ is given by a series in $x \equiv \sigma_i^2(q, z)$,

$$K(x) = \sum_{L=1}^\infty C_L x^{L-1}$$

Padé ansatz

$$K_{nm}^{\text{pade}}(x) \equiv \frac{1 + \sum_{i=1}^n a_i x^i}{1 + \sum_{j=1}^m b_j x^j}$$

Match for small $x$, using coeff. up to three loop $C_1 = 1$, $C_2 \approx 0.71$, $C_3 \approx 1.05$

- Two-loop matching (using $C_1$, $C_2$): $n, m = 0, 1$
- Three-loop matching: either $n, m = 0, 2$ or $n, m = 1, 1$
Result for Padé resummed small-k limit

Correction to $P(k,z)$ rel. to one–loop

$P(k,z)/P_{1\text{-}\text{loop}}(k,z)$

redshift $z$

black=SPT, solid=Padé resummed result

Mathias Garny (DESY Hamburg / CERN)
Padé improved PT

\[ P(k, z) = P_{\text{lin}}(k, z) + P_{\text{pade}}^{\text{small} - k}(k, z) \]
\[ + P_{1-\text{loop}}^{\text{sub}}(k, z) + P_{2-\text{loop}}^{\text{sub}}(k, z) + P_{3-\text{loop}}^{\text{sub}}(k, z) + \ldots , \]

where

\[ P_{L-\text{loop}}^{\text{sub}}(k, z) \equiv P_{L-\text{loop}}(k, z) - P_{L-\text{loop}}^{\text{small} - k}(k, z) \]
Padé improved PT vs N-body

\[ z = 0.375 \]

black=SPT, blue=Padé improved PT, red=N-body Horizon Run 2
Padé improved PT vs N-body

$z = 0$

black=SPT, blue=Padé improved PT, red=N-body Horizon Run 2

On the non-linear scale of cosmological perturbation theory

Mathias Garny (DESY Hamburg / CERN)
Conclusion

- Three loop larger than one loop at $z = 0$ at all scales
- Expected for Eisenstein-Hu-like spectrum
- Loop expansion exhibits behaviour of asymptotic series
- Padé resummation in small $k$ limit
- Improved perturbative expansion with better convergence properties at BAO scales, good agreement with N-body data
Basic formalism for large scale structure

- Density contrast $\rho(x, \tau) = \bar{\rho}(\tau)(1 + \delta(x, \tau))$, pec. velocity $u(x, \tau)$
- 1st and 2nd moment of Vlasov eq. for $f(x, p, \tau)$, neglect multi-streaming (pressure, stress/viscosity $\rightarrow$ talks by Mercolli, Zaldarriaga)

\[
\frac{\partial \delta(x, \tau)}{\partial \tau} + \nabla \cdot \{(1 + \delta(x, \tau)u(x, \tau)\} = 0 \quad \text{(continuity)}
\]
\[
\frac{\partial u(x, \tau)}{\partial \tau} + H u + u \cdot \nabla u = -\nabla \Phi \quad \text{(Euler)}
\]
\[
\nabla^2 \Phi(x, \tau) = \frac{3}{2} \Omega_m H^2 \delta(x, \tau) \quad \text{(Poisson)}
\]

- neglect vorticity $\nabla \times u \approx 0 \Rightarrow$ sufficient to use $\theta = \nabla \cdot u$
- solution of linearized eqs ($D_+ \sim a$, $D_- \sim a^{-3/2}$ in EdS)

\[
\delta_{\text{lin}}(x, \tau) = D_+(\tau)\delta_0(x) + O(D_-)
\]

- Power spectrum in Fourier space

\[
\langle \delta(k, \tau)\delta(k', \tau) \rangle = \delta^{(3)}(k + k')P(k, \tau)
\]

assume Gaussian IC described by $P_0(k) = P(k, \tau_0)$
Expansion parameter

\[ \sigma_i^2(k, z) \equiv 4\pi \int_0^k dq \, q^2 P_{\text{lin}}(q, z) \]

Large \( k \)

\[ P_{1-\text{loop}}(k) \sim \left( 1.14P_{\text{lin}}(k, z) - 0.55k\partial_k P_{\text{lin}}(k, z) + 0.1[k\partial_k]^2 P_{\text{lin}}(k, z) \right) \sigma_i^2(k, z) \]

\[ P_{2-\text{loop}}(k) \sim \left( 2.14P_{\text{lin}}(k, z) - 1.62k\partial_k P_{\text{lin}}(k, z) + 0.55[k\partial_k]^2 P_{\text{lin}}(k, z) \right. \]

\[ - 0.082[k\partial_k]^3 P_{\text{lin}}(k, z) + 0.005[k\partial_k]^4 P_{\text{lin}}(k, z) \right) \sigma_i^4(k, z) \]

Small \( k \)

\[ P_{1-\text{loop}}(k) \rightarrow -\frac{61}{105} k^2 P_{\text{lin}}(k, z) \frac{4\pi}{3} \int_0^\infty dq P_{\text{lin}}(q, z) \]

\[ P_{2-\text{loop}}(k) \rightarrow -\frac{44764}{143325} k^2 P_{\text{lin}}(k, z) \frac{4\pi}{3} \int_0^\infty dq P_{\text{lin}}(q, z) J(q) \]

where \( J(q) = 4\pi \int_0^q dp \, p^2 g(p/q) P_{\text{lin}}(p, z) \sim \sigma_i^2(q, z) \)
Integrand kernel \( k P_{\text{lin}}(k) K_L(\sigma^2_i(k, z)) \) for the power spectrum as obtained in SPT at one-loop (black dashed), two loops (black dot-dashed), three loops (black dotted). The solid lines are the integrand kernels obtained after Padé resummation, \( K^{pade}_{01} \) (green), \( K^{pade}_{02} \) (blue) and \( K^{pade}_{11} \) (magenta).
On the non-linear scale of cosmological perturbation theory

---

Mathias Garny (DESY Hamburg / CERN)
Enhancement from soft loops $q_i \ll k$

- Scale $\sigma_d$ related to $F_n \propto k \cdot q_i/q_i^2$ for soft $q_i \ll k$

$$k^2 \sigma_d^2(z) \equiv \int d^3q \frac{(k \cdot q)^2}{q^4} P_{\text{lin}}(q, z) = \frac{4\pi}{3} k^2 \int dq P_{\text{lin}}(q, z)$$

- Power spectrum at 1-loop, for large $k$

$$P_{22} \to k^2 \sigma_d^2(z) P_{\text{lin}}(k, z), \quad 2P_{13} \to -k^2 \sigma_d^2(z) P_{\text{lin}}(k, z)$$

- At L-loop $\sim (k^2 \sigma_d^2)^\ell$ with $\ell \leq L$ (resummation $\to$ RPT \textit{Crocce, Scoccimarro 05})

- Leading terms ($\ell = L$) cancel when summing over all contributions \textit{Bertschinger, Jain 95}

- Subleading terms cancel as well ($1 \leq \ell \leq L$) \textit{Blas, MG, Konstandin 13}

- Expected from Galilean invariance \textit{Frieman,Scoccimarro 95; Pietroni,Peloso 13}

$\Rightarrow$ SPT as good as RPT for equal-time correlators \textit{Sugiyama, Spergel 13}

- Cancellation can be made manifest for loop integrand \textit{Blas, MG, Konstandin 13; Carrasco, Foreman, Green, Senatore 13}

$\Rightarrow$ very helpful for Monte-Carlo integration