On the non-linear scale of cosmological perturbation theory

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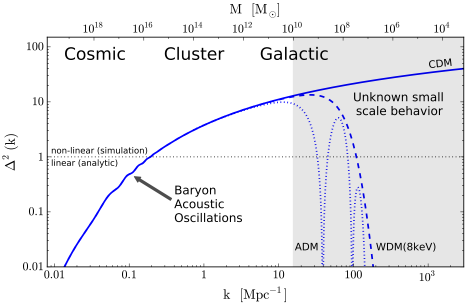
Cosmo Cambridge, 02.09.13

based on 1304.1546 and 1309.xxxx
with Diego Blas, Thomas Konstandin

Motivation

- Matter power spectrum in the regime of baryon acoustic oscillations will be measured with high precision (Euclid, . . .)
- Desirable to develop a (fast) method for theoretical prediction for a given set of parameters (+ analytic understanding of onset of non-linearities)
- Weakly non-linear regime ⇒ borderline for perturbation theory



$\Delta^2(k,z) = 4\pi k^3 P(k,z)$

Kuhlen, Vogelsberger, Angulo 1209.5745

Outline

- Standard Perturbation Theory
- Three loop result
- Padé resummation
- Padé improved PT

Standard Perturbation Theory

- Poisson/Euler/Continuity eq. for density contrast $\delta = \rho/\bar{\rho} - 1$ and pec. velocity (neglect vorticity/viscosity $\rightarrow$ talks by Mercolli, Zaldarriaga)
- Expand solution in $\delta_0(\mathbf{k}) = \delta(\mathbf{k}, z_0 \sim 10^3)$, for EdS / growing mode

$$\begin{aligned}\delta(\mathbf{k}, z) &= \sum_{n=1}^{\infty} (D_+(z))^n \int_{q_i} \delta^{(3)}(\mathbf{k} - \sum \mathbf{q}_i) F_n(\mathbf{q}_1, \ldots, \mathbf{q}_n) \delta_0(\mathbf{q}_1) \cdots \delta_0(\mathbf{q}_n) \\ &= D_+(z)\delta_0(\mathbf{k}) + \ldots\end{aligned}$$

e.g. $F_2^s(\mathbf{q}_1, \mathbf{q}_2) = \frac{5}{7} + \frac{1}{2}\frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{q_1 q_2}\left(\frac{q_1}{q_2} + \frac{q_2}{q_1}\right) + \frac{2}{7}\frac{(\mathbf{q}_1 \cdot \mathbf{q}_2)^2}{q_1^2 q_2^2}$

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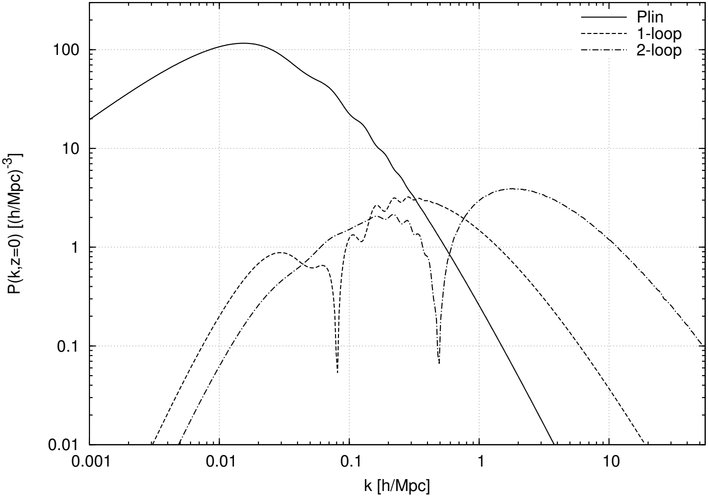
- Power spectrum $\langle \delta(\mathbf{k}, z)\delta(\mathbf{k}', z)\rangle = \delta^{(3)}(\mathbf{k} + \mathbf{k}')P(k, z)$ is obtained using Wick theorem in terms of $P_0(q) = P(q, z_0)$ for Gaussian IC

$$\begin{aligned}P(k, z) &= \overbrace{D_+(z)^2 P_0(k)}^{P_{lin}} + \overbrace{D_+(z)^4 (2P_{13} + P_{22})}^{P_{1-loop}} \\ &\quad + D_+(z)^6 (2P_{15} + 2P_{24} + P_{33}) + \ldots\end{aligned}$$

e.g. $P_{22}(k) = 2\int d^3q\, F_2^s(\mathbf{q}, \mathbf{k} - \mathbf{q})^2\, P_0(q) P_0(|\mathbf{k} - \mathbf{q}|)$

Standard Perturbation Theory



SPT breaks down at small scales / late times $[P_{L-loop} \propto D_+^{2L+2} \sim \left(\frac{1}{1+z}\right)^{2L+2}]$

initial spectrum from CAMB (ΛCDM - WMAP5)

Standard Perturbation Theory

- Many approaches have been developed to improve behaviour at large k (closure, LPT, RPT, eikonal, ...) *see e.g. Carlson, White, Padmanabhan 0905.0479*
- Enhanced contributions ($\propto k^{2n}$, $n \leq L$) related to interaction with soft modes cancel out completely in equal-time correlators (like the PS), as expected from Galilean invariance

  see e.g. Bertschinger, Jain 95; Frieman, Scoccimarro 95; Anselmi, Pietroni 12; Pietroni, Peloso 13; Sugiyama, Futamase 13; Blas, MG, Konstandin 13; Carrasco, Foreman, Green, Senatore 13; . . .

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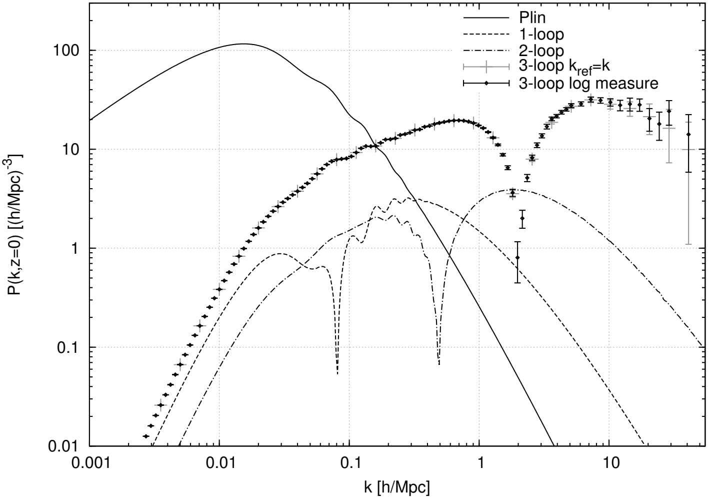
⇒ RPT 'as good as' SPT for large-k for equal-time correlators (but very useful for unequal-time corr. as e.g. propagator) *e.g. Sugiyama, Spergel 13*

- Cancellations (leading + sub-leading terms) can be made manifest for loop integrand

  Blas, MG, Konstandin 13; Carrasco, Foreman, Green, Senatore 13

⇒ crucial for Monte-Carlo integration

Three loop



Diego Blas, MG, Thomas Konstandin 1309.xxxx (power spectrum); see also Bernardeau, Taruya, Nishimichi, 1211.1571 (propagator)

Loop expansion of PS in the limit of small k

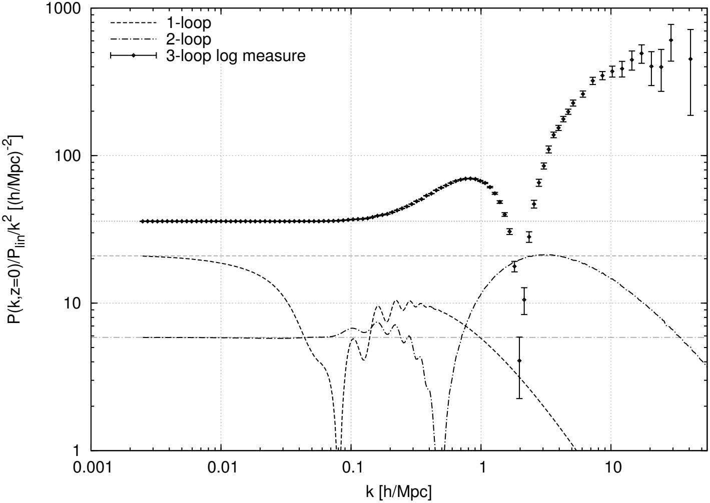
For small k *Fry AJ421 (1994) 21*

$$P_{L-loop}(k,z) \quad \to \quad \frac{(2L+1)!}{2^{L-1}L!}\, P_{lin}(k,z) \int_{q_1} \cdots \int_{q_L} F^s_{2L+1}(\mathbf{k}, \mathbf{q}_1, -\mathbf{q}_1, \ldots, \mathbf{q}_L, -\mathbf{q}_L) \times P_{lin}(q_1,z) \cdots P_{lin}(q_L,z)$$

Using property $F^s_{2L+1} \propto k^2$ for $k \ll q_i$ *Goroff, Grinstein, Rey, Wise AJ311 (1986) 6*

$$P_{L-loop}(k,z) \propto k^2 P_{lin}(k,z) \qquad \text{for small } k$$

One, two and three loop normalized to $k^2 P_{lin}(k,z)$



$P_{L-loop}(k,z) \propto k^2 P_{lin}(k,z)$

up to $k = 0.003, 0.06, 0.08$ h/Mpc for $1, 2, 3$-loop at %-level

Loop expansion of PS in the limit of small k

Using property $F^s_{2L+1} \propto k^2/q^2$ for $k \ll q_i$ and $q =$max(q_i)

$$P_{L-loop} \to P^{small-k}_{L-loop} = -\frac{244\pi}{325}\, k^2 P_{lin}(k,z) \times C_L \times \int_0^\infty dq P_{lin}(q,z)\, \sigma_l^{2L-2}(q,z)$$

with coeff. C_L ($C_1 = 1$, $C_2 \simeq 0.71$, $C_3 \simeq 1.05$) and scale-dep. variance

$$\sigma_l^2(q,z) \equiv 4\pi \int_0^q dp\, p^2 P_{lin}(p,z)$$

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Estimate for Eisenstein-Hu spectrum with $n_s \simeq 1$

$$P^{small-k}_{L-loop} \propto k^2 P_{lin}(k,z) \times C_L \times \frac{(3L-1)!}{2^{3L}} D_+(z)^{2L}$$

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- Terms decrease up to a certain order $L_{max}(z)$, then increase
- Typical behaviour of an asymptotic series (e.g. loop exp. in QED)
- Partial sum up to smallest term yields best result, with error of order the smallest term (e.g. $P_{2-loop}/P_{lin} \simeq 6\%$ at $z = 0$, $k = 0.1h/\text{Mpc}$)

Padé ansatz

Goal: improve convergence to go to %-accuracy
Idea: resummation in small-k limit

$$P_{small-k}(k,z) = -\frac{244\pi}{315}\, k^2 P_{lin}(k,z) \times \int_0^\infty dq P_{lin}(q,z)\, K(\sigma_l^2(q,z))$$

where the integrand kernel K is given by a series in $x \equiv \sigma_l^2(q,z)$,

$$K(x) = \sum_{L=1}^{\infty} C_L x^{L-1}$$

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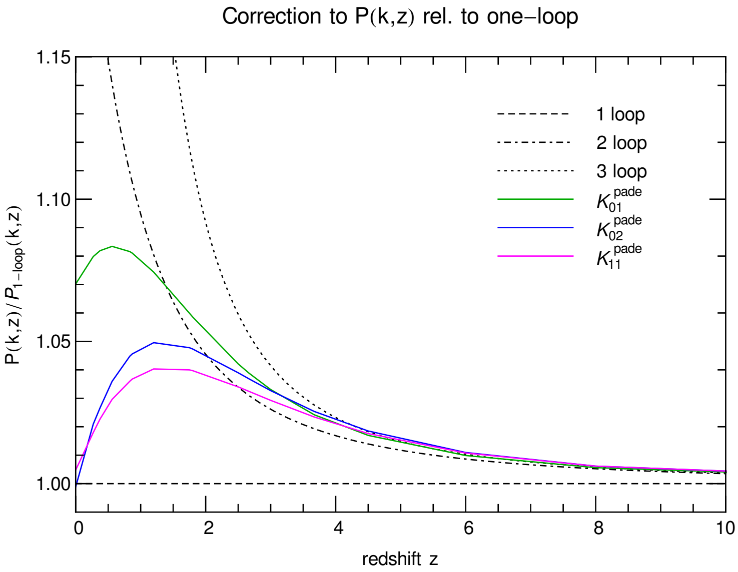
Padé ansatz

$$K_{nm}^{pade}(x) \equiv \frac{1 + \sum_{i=1}^{n} a_i x^i}{1 + \sum_{j=1}^{m} b_j x^j}$$

Match for small x, using coeff. up to three loop $C_1 = 1$, $C_2 \simeq 0.71$, $C_3 \simeq 1.05$

- Two-loop matching (using C_1, C_2): $n, m = 0, 1$
- Three-loop matching: either $n, m = 0, 2$ or $n, m = 1, 1$

Result for Padé resummed small-k limit



black=SPT, solid=Padé resummed result

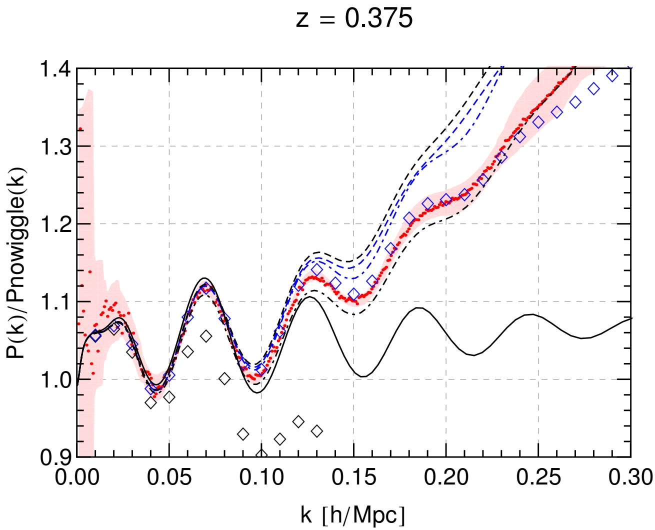
Padé improved PT

$$P(k,z) = P_{lin}(k,z) + P^{pade}_{small-k}(k,z) + P^{sub}_{1-loop}(k,z) + P^{sub}_{2-loop}(k,z) + P^{sub}_{3-loop}(k,z) + \dots ,$$

where

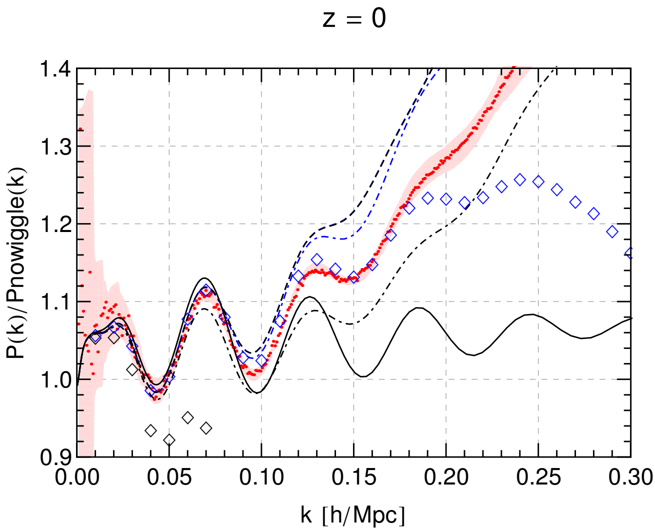
$$P^{sub}_{L-loop}(k,z) \equiv P_{L-loop}(k,z) - P^{small-k}_{L-loop}(k,z)$$

Padé improved PT vs N-body



black=SPT, blue=Padé improved PT, red=N-body Horizon Run 2

Padé improved PT vs N-body



black=SPT, blue=Padé improved PT, red=N-body Horizon Run 2

Conclusion

- Three loop larger than one loop at $z = 0$ at all scales
- Expected for Eisenstein-Hu-like spectrum
- Loop expansion exhibits behaviour of asymptotic series
- Padé resummation in small k limit
- Improved perturbative expansion with better convergence properties at BAO scales, good agreement with N-body data

Basic formalism for large scale structure

- Density contrast $\rho(\mathbf{x},\tau) = \bar{\rho}(\tau)(1+\delta(\mathbf{x},\tau))$, pec. velocity $\mathbf{u}(\mathbf{x},\tau)$
- 1st and 2nd moment of Vlasov eq. for $f(\mathbf{x},p,\tau)$, neglect multi-streaming (pressure, stress/viscosity $\rightarrow$ talks by Mercolli, Zaldarriaga)

$$\begin{aligned} \frac{\partial\delta(\mathbf{x},\tau)}{\partial\tau} + \nabla\cdot\{(1+\delta(\mathbf{x},\tau)\mathbf{u}(\mathbf{x},\tau)\} &= 0 \quad \text{(continuity)} \\ \frac{\partial\mathbf{u}(\mathbf{x},\tau)}{\partial\tau} + \mathcal{H}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} &= -\nabla\Phi \quad \text{(Euler)} \\ \nabla^2\Phi(\mathbf{x},\tau) &= \frac{3}{2}\Omega_m\mathcal{H}^2\delta(\mathbf{x},\tau) \quad \text{(Poisson)} \end{aligned}$$

- neglect vorticity $\nabla\times\mathbf{u}\approx 0 \Rightarrow$ sufficient to use $\theta = \nabla\cdot\mathbf{u}$
- solution of linearized eqs ($D_+ \sim a, D_- \sim a^{-3/2}$ in EdS)

$$\delta_{lin}(\mathbf{x},\tau) = D_+(\tau)\delta_0(\mathbf{x}) + \mathcal{O}(D_-)$$

- Power spectrum in Fourier space

$$\langle\delta(\mathbf{k},\tau)\delta(\mathbf{k}',\tau)\rangle = \delta^{(3)}(\mathbf{k}+\mathbf{k}')P(k,\tau)$$

assume Gaussian IC described by $P_0(k) = P(k,\tau_0)$

Expansion parameter

$$\sigma_l^2(k,z) \quad \equiv \quad 4\pi \int_0^k dq\, q^2 P_{lin}(q,z)$$

Large k

$$P_{1-loop}(k) \quad \sim \quad \Big(1.14 P_{lin}(k,z) - 0.55 k\partial_k P_{lin}(k,z) + 0.1[k\partial_k]^2 P_{lin}(k,z)\Big)\, \sigma_l^2(k,z)$$

$$P_{2-loop}(k) \quad \sim \quad \Big(2.14 P_{lin}(k,z) - 1.62 k\partial_k P_{lin}(k,z) + 0.55[k\partial_k]^2 P_{lin}(k,z) - 0.082[k\partial_k]^3 P_{lin}(k,z) + 0.005[k\partial_k]^4 P_{lin}(k,z)\Big)\sigma_l^4(k,z)$$

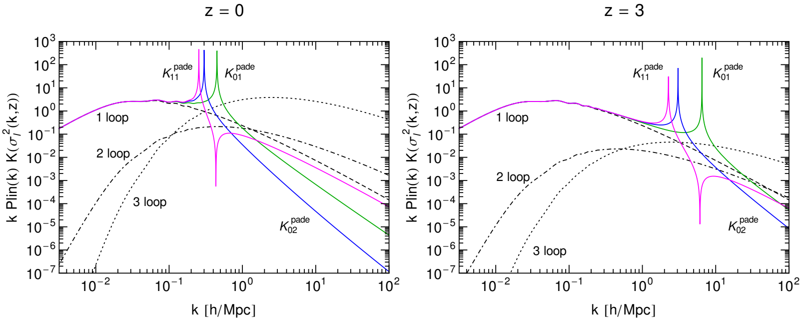
Small k

$$P_{1-loop}(k) \quad \to \quad -\frac{61}{105} k^2 P_{lin}(k,z) \frac{4\pi}{3} \int_0^\infty dq P_{lin}(q,z)$$

$$P_{2-loop}(k) \quad \to \quad -\frac{44764}{143325} k^2 P_{lin}(k,z) \frac{4\pi}{3} \int_0^\infty dq P_{lin}(q,z) J(q)$$

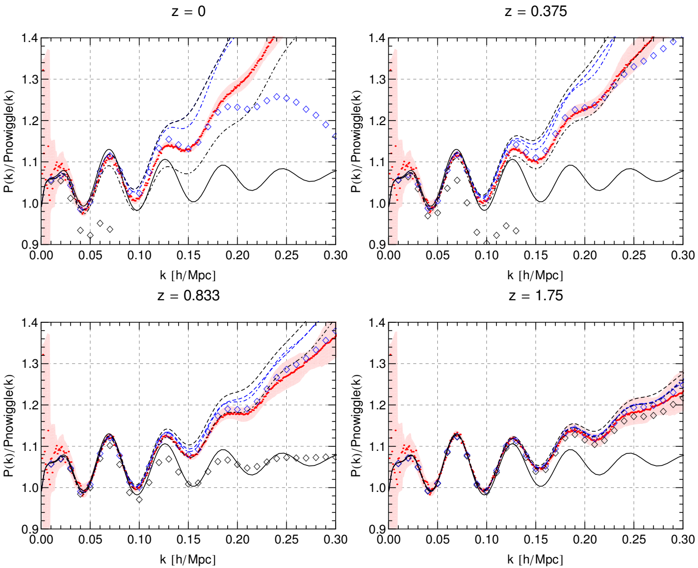
where $J(q) = 4\pi \int_0^q dp\, p^2 g(p/q) P_{lin}(p,z) \sim \sigma_l^2(q,z)$

Result for Padé resummed small-k limit



Integrand kernel $k\,P_{lin}(k)K_L(\sigma_l^2(k,z))$ for the power spectrum as obtained in SPT at one-loop (black dashed), two loops (black dot-dashed), three loops (black dotted). The solid lines are the integrand kernels obtained after Padé resummation, K_{01}^{pade} (green), K_{02}^{pade} (blue) and K_{11}^{pade} (magenta).

Padé improved PT vs N-body



Enhancement from soft loops $q_i \ll k$

- Scale σ_d related to $F_n \propto k \cdot q_i / q_i^2$ for soft $q_i \ll k$

$$k^2 \sigma_d^2(z) \quad \equiv \quad \int d^3q \frac{(k \cdot q)^2}{q^4} P_{lin}(q,z) = \frac{4\pi}{3} k^2 \int dq\, P_{lin}(q,z)$$

- Power spectrum at 1-loop, for large k

$$P_{22} \to k^2 \sigma_d^2(z) P_{lin}(k,z), \quad 2P_{13} \to -k^2 \sigma_d^2(z) P_{lin}(k,z)$$

- At L-loop $\sim (k^2 \sigma_d^2)^\ell$ with $\ell \leq L$ (resummation $\to$ RPT *Crocce, Scoccimarro 05*)
- Leading terms ($\ell = L$) cancel when summing over all contributions *Bertschinger, Jain 95*
- Subleading terms cancel as well ($1 \leq \ell \leq L$) *Blas, MG, Konstandin 13*
- Expected from Galilean invariance *Frieman,Scoccimarro 95; Pietroni,Peloso 13*

$\Rightarrow$ SPT as good as RPT for equal-time correlators *Sugiyama, Spergel 13*

- Cancellation can be made manifest for loop integrand *Blas, MG, Konstandin 13; Carrasco, Foreman, Green, Senatore 13*

$\Rightarrow$ very helpful for Monte-Carlo integration