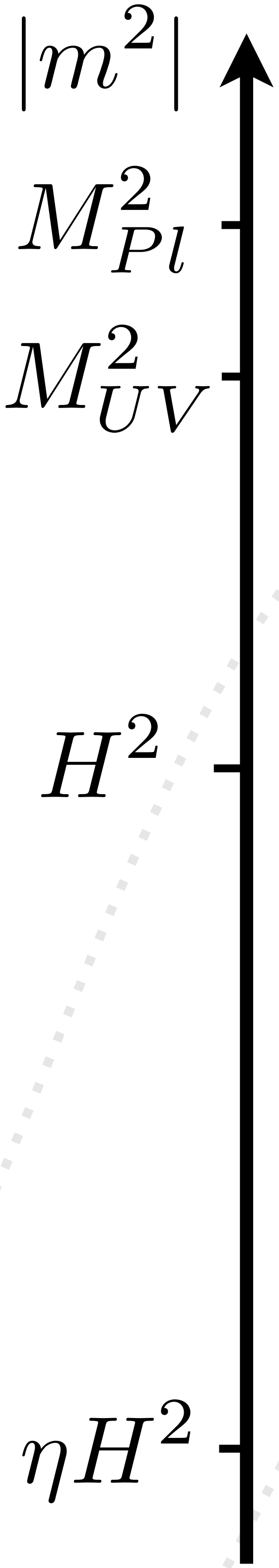
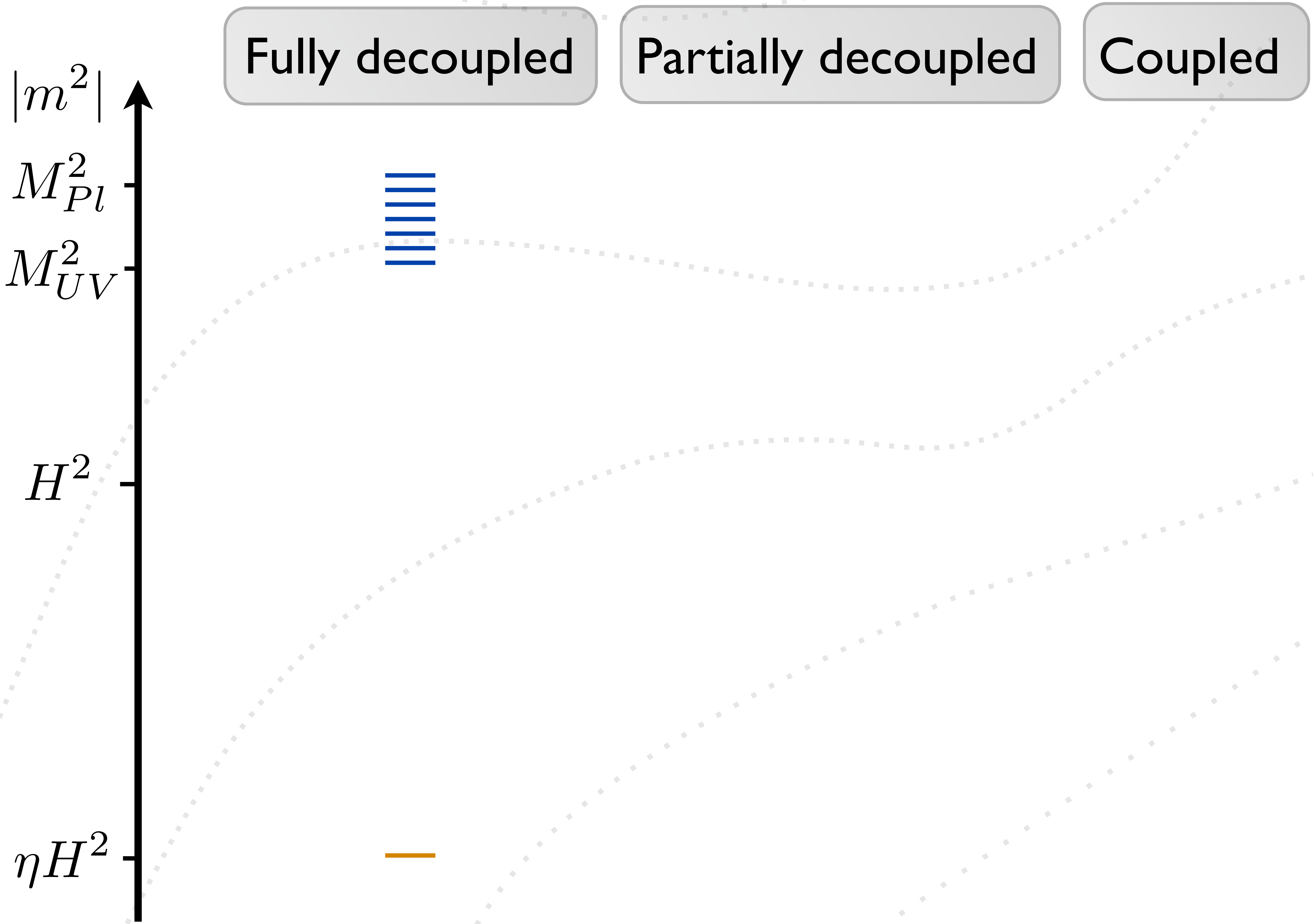
# Charting an Inflationary Landscape with Random Matrix Theory

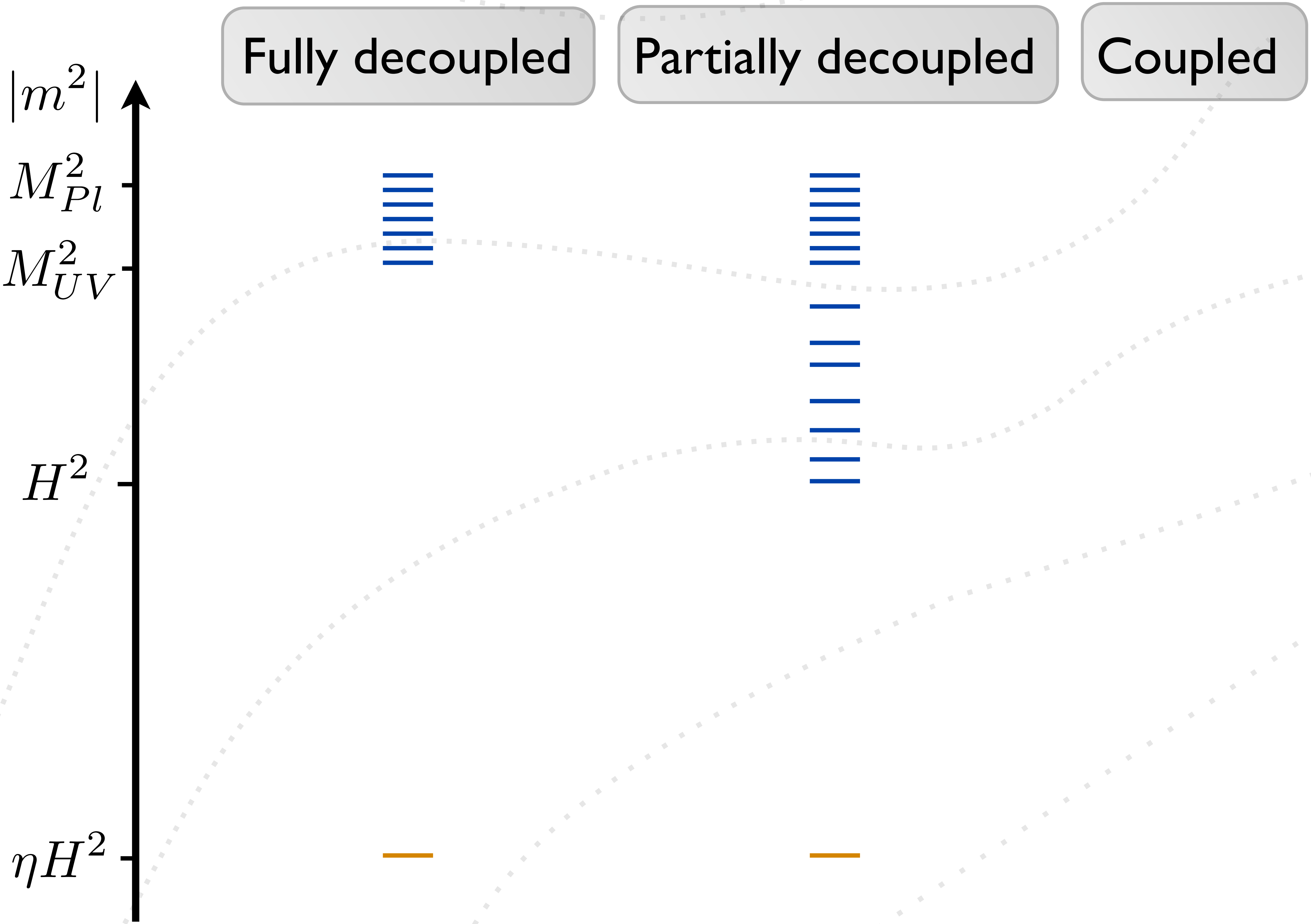

*M.C. David Marsh*
*University of Oxford*


Based on: arXiv:1307:3559,
with L. McAllister, E. Pajer and T. Wrase.




*There is no a priori reason to expect inflation to be driven by a single scalar field.*


Fully decoupled
Partially decoupled
Coupled
$|m^2|$
$M_{Pl}^2$
$M_{UV}^2$
$H^2$
$\eta H^2$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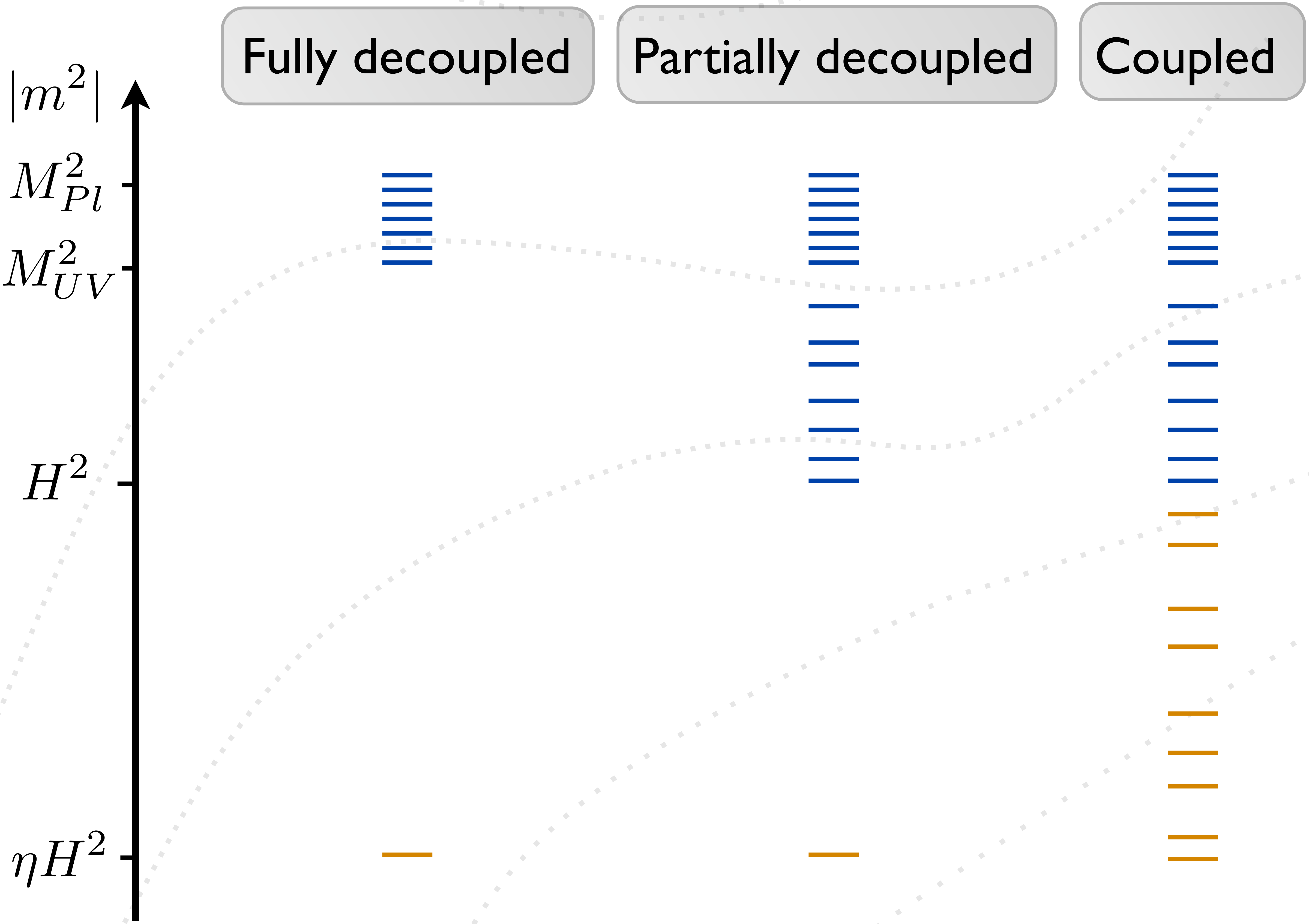

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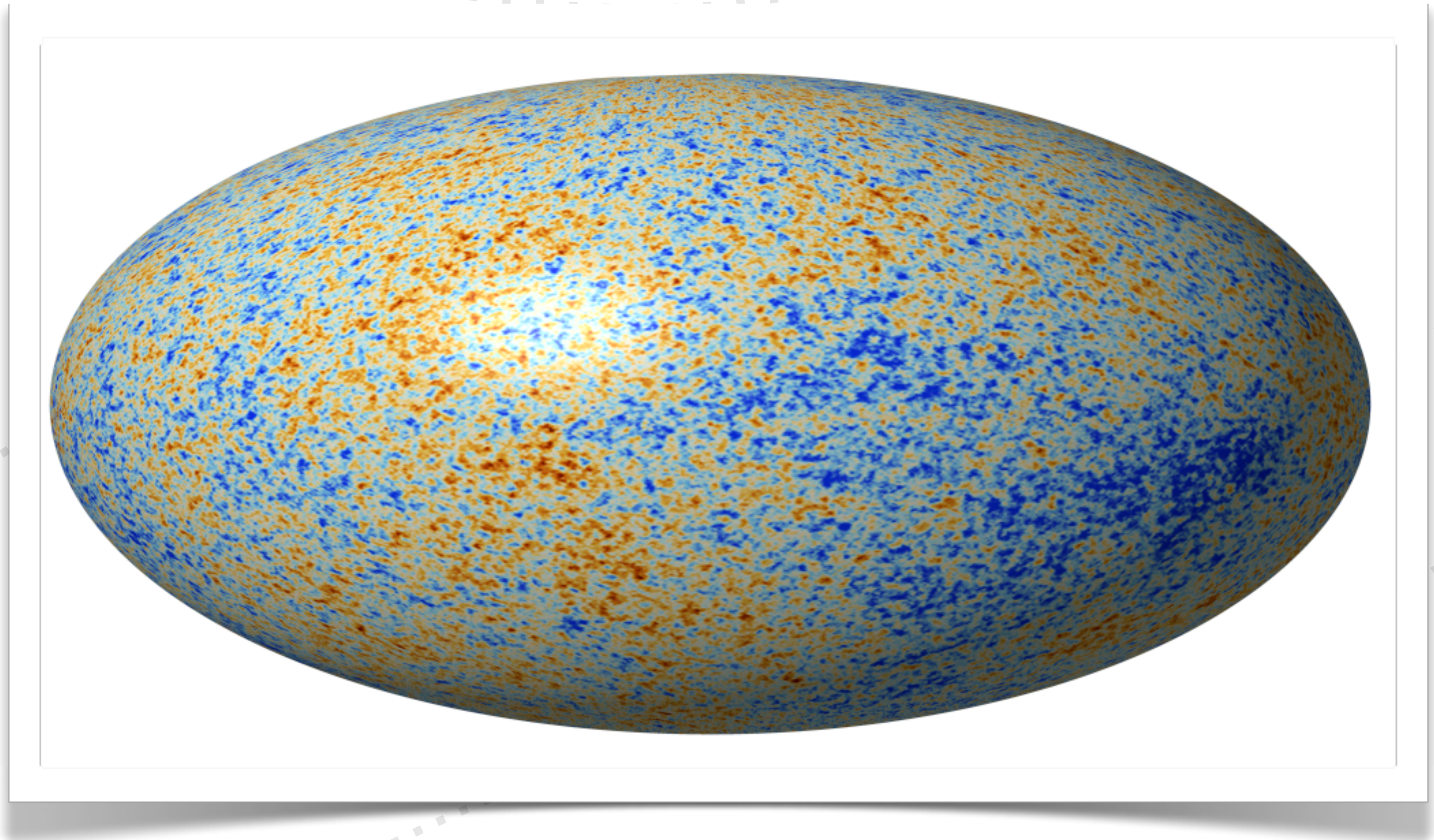
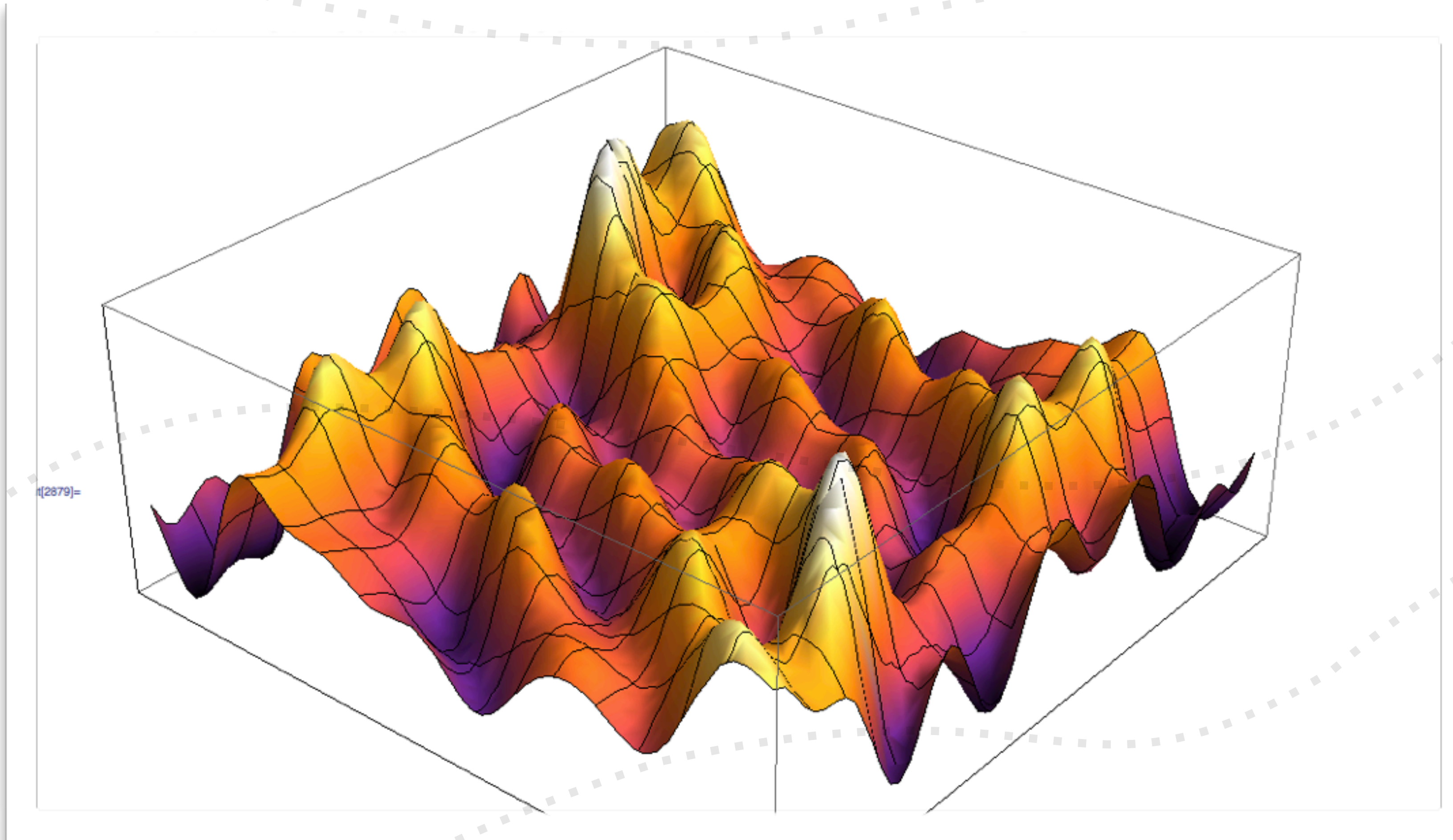


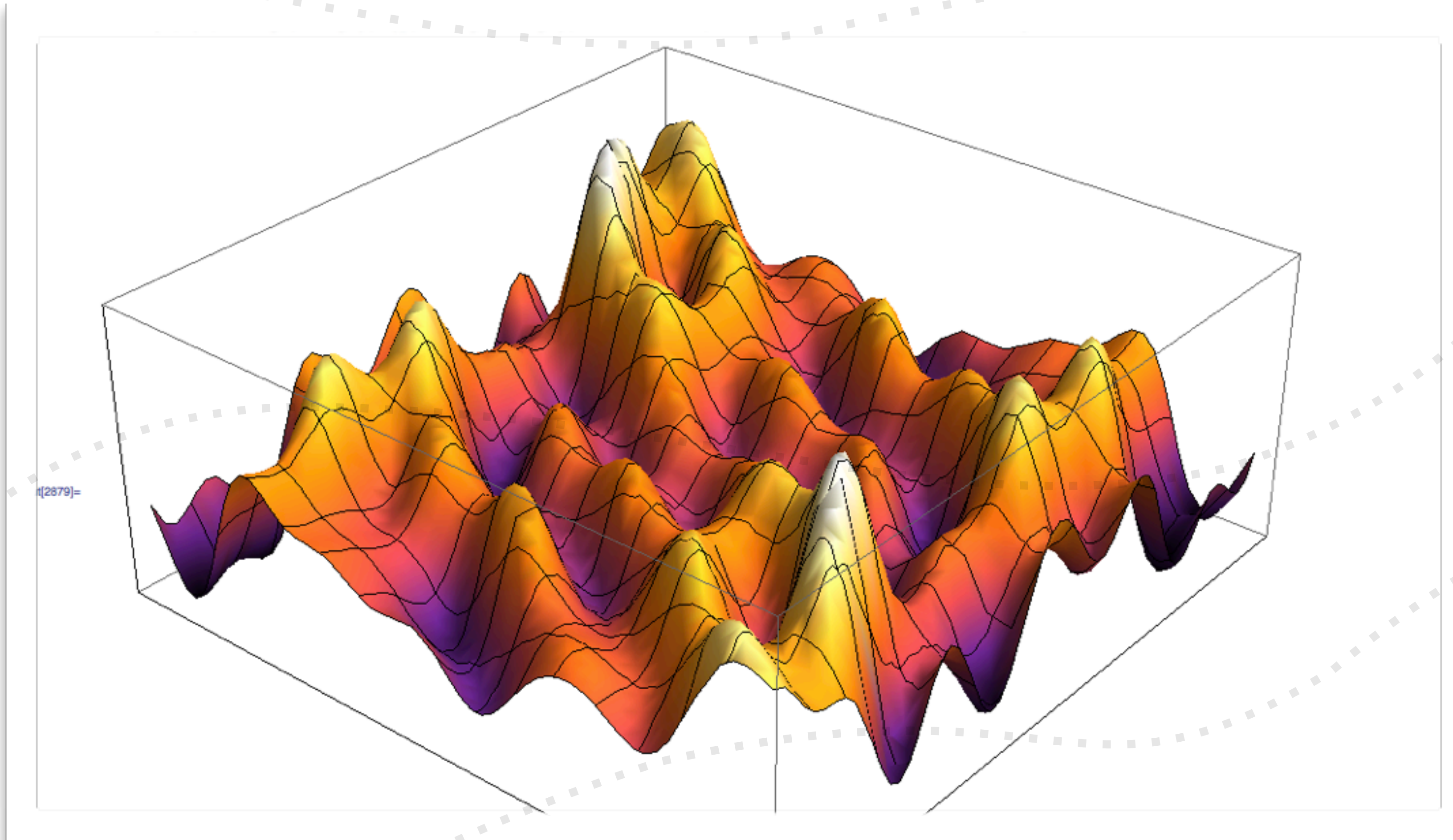
Yet, the Planck analysis of CMB data appear to reveal no signs of multifield dynamics through primordial isocurvature or local non-Gaussianities.

In order to understand the severity of these constraints, a better understanding of the *typical predictions* of multifield inflation would be illuminating.



One way to address this question statistically is to create large ensembles of multifield scalar potentials, and to study the resulting dynamics for some set of initial conditions.

Traditionally, this approach has been applied with some success to relatively small systems with *$N<10$* fields, *c.f. talks by Battefeld, Frazer, ...*

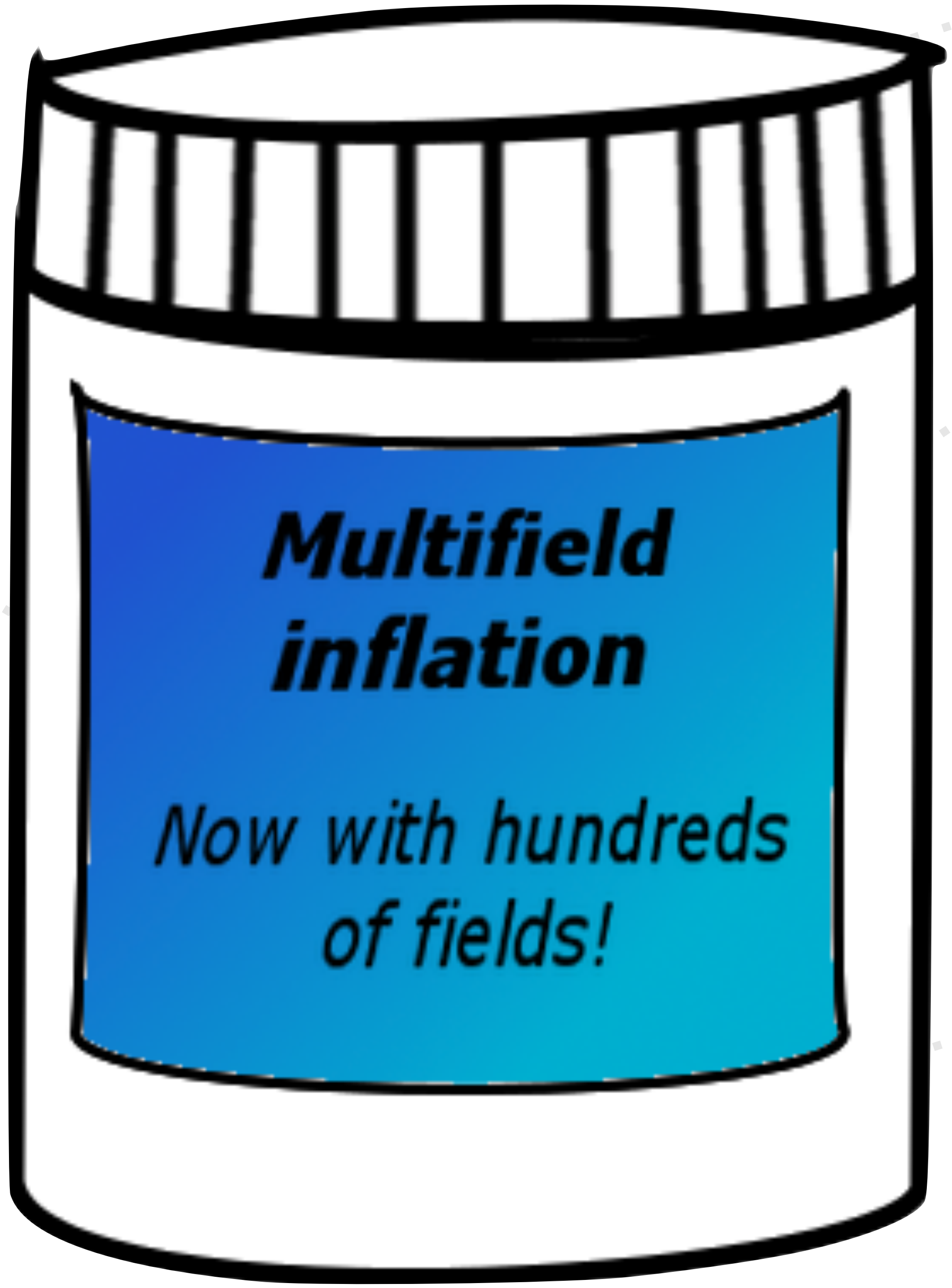


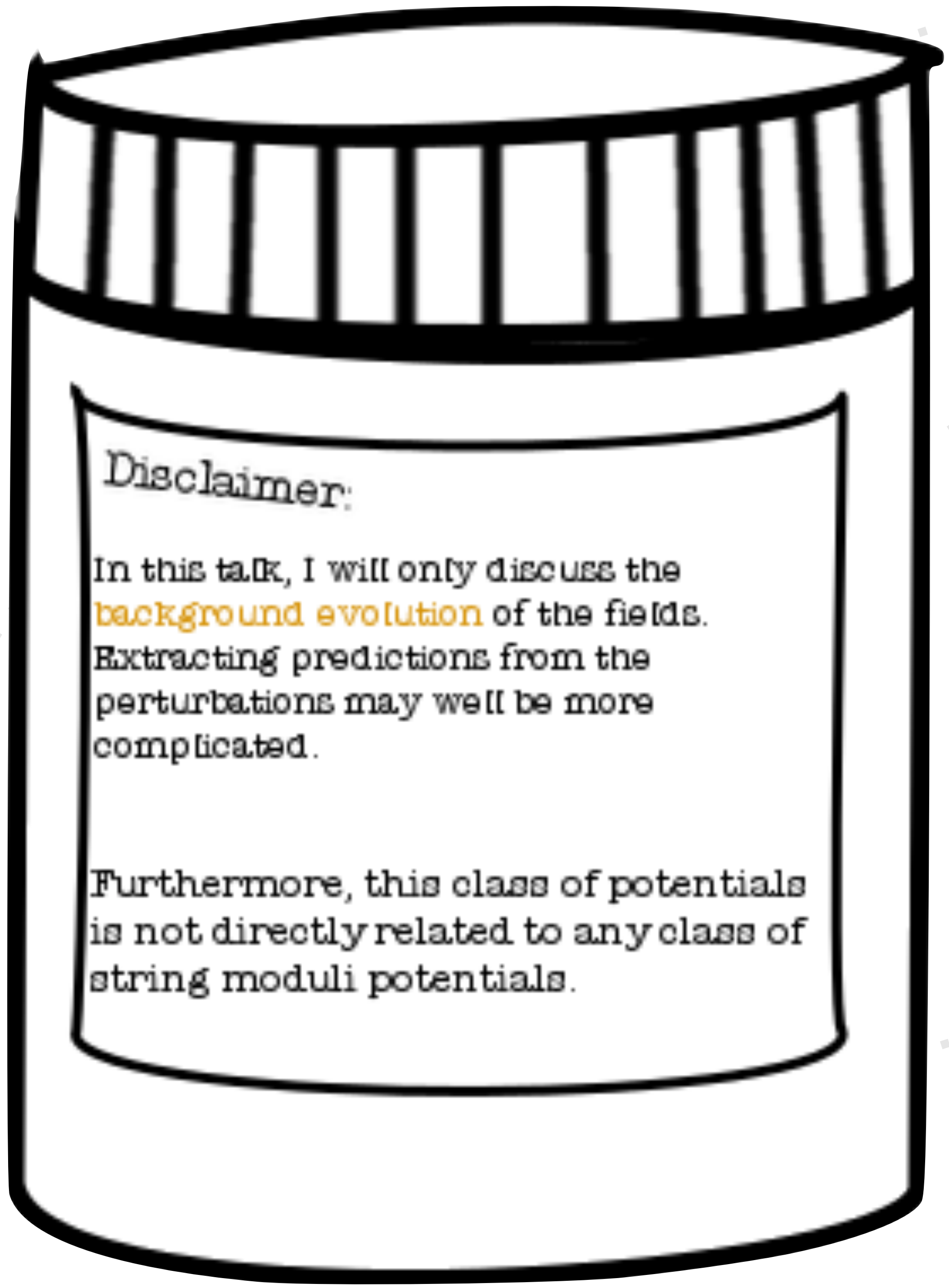
***Computational cost:***

Random Fourier Potential: $(k_{\rm max}/k_{\rm min})^N$ .

New method presented here: $\sim N^3$ .

This new method of constructing ensembles of random scalar potentials allows us to study systems with hundreds of fields.


Multifield inflation
Now with hundreds of fields!

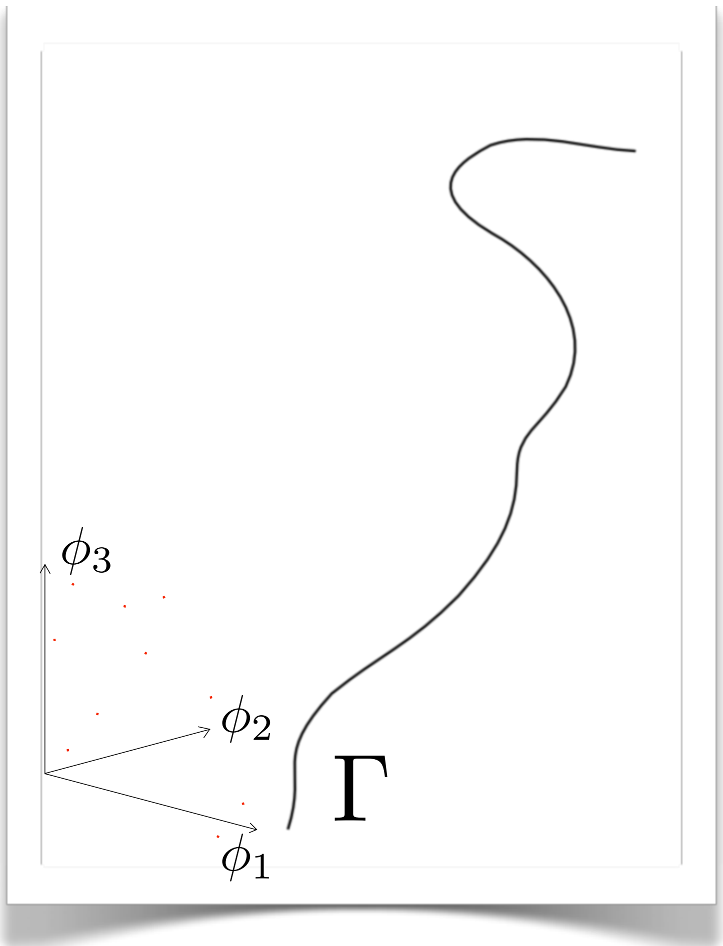

Disclaimer:
In this talk, I will only discuss the background evolution of the fields. Extracting predictions from the perturbations may well be more complicated.
Furthermore, this class of potentials is not directly related to any class of string moduli potentials.

***Large N universality:***
Many physical systems exhibit emergent simplicity in the $N \to \infty$ limit.

There is no a priori guarantee that the aspects relevant for inflation will be universal, but I will argue that some predictions indeed appear to be so.

# *A novel, local approach to multifield potentials*




Take $(\phi_1, \ldots, \phi_N)$ to be fields in Euclidean $\mathbb{R}^N$ for which we want to construct an ensemble of potentials which, on average, treats all fields as equals.

*Our new approach* is based on constructing potentials *locally around a path* $\Gamma$ in field space.

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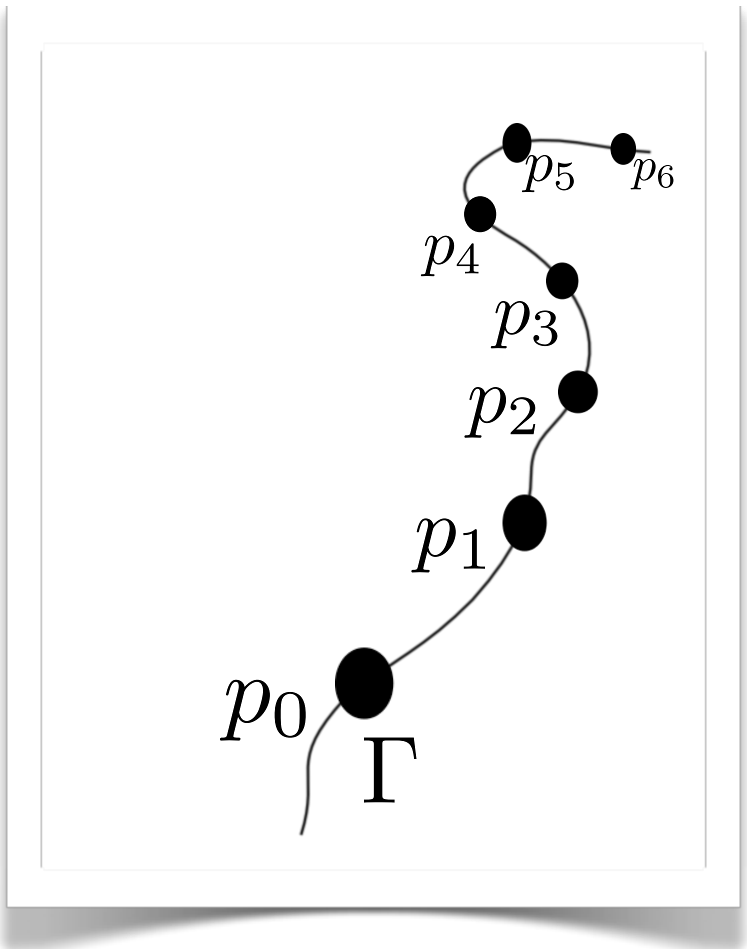


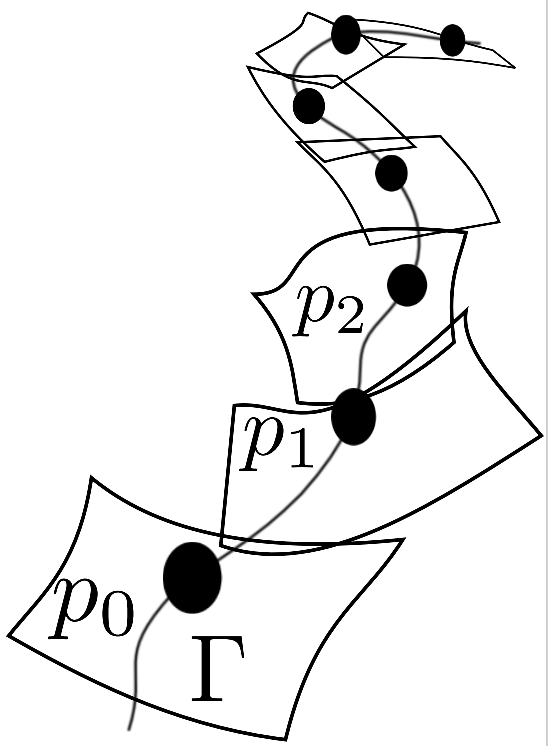

Take $(\phi_1, \ldots, \phi_N)$ to be fields in Euclidean $\mathbb{R}^N$ for which we want to construct an ensemble of potentials which, on average, treats all fields as equals.

*Our new approach* is based on constructing potentials *locally around a path* $\Gamma$ in field space.

Consider first a string of nearby points on $\Gamma$, and specify

$V\big|p_0, \ \partial_a V\big|p_0,$ and $\partial_{ab} V\big|p_0 = \mathcal{H}\big|_{p_0}$ .

# *A novel, local approach to multifield potentials*




From the value of the potential, gradient and Hessian at $p_0$, the value of the potential and gradient at $p_1$ may be obtained to leading order in Taylor expansion:

$$V\big|p_1 = V\big|p_0 + \delta\phi^a \partial_a V\big|p_0 \,,$$

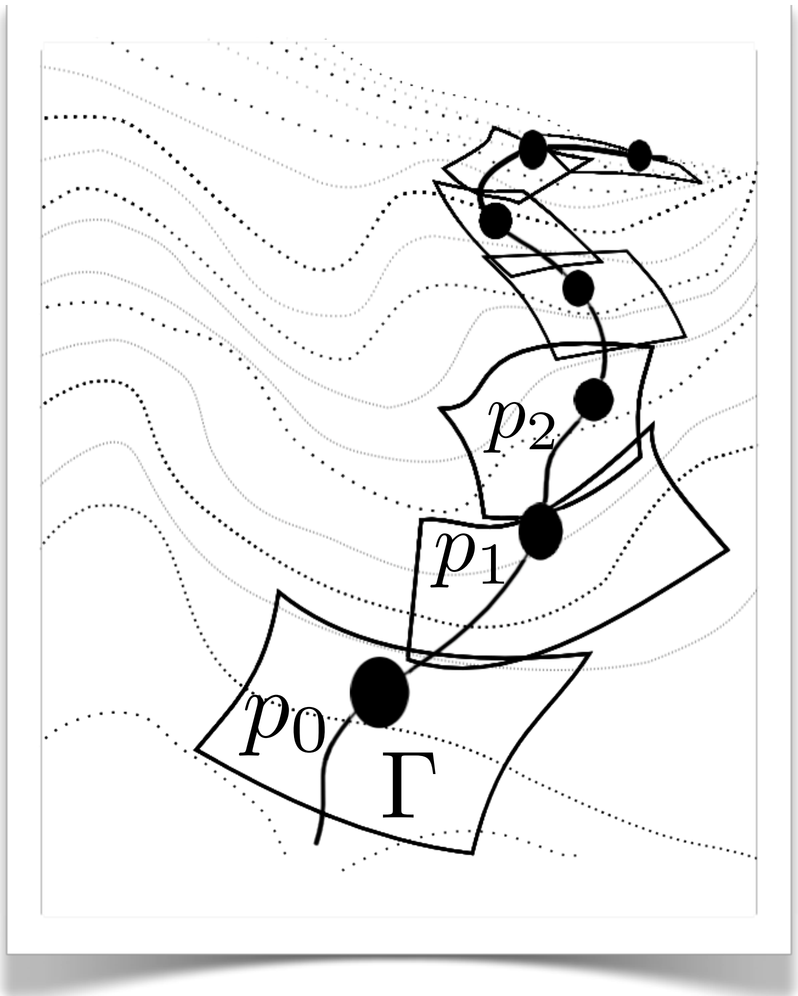
$$\partial_a V\big|p_1 = \partial_a V\big|p_0 + \delta\phi^b \partial^2_{ab} V\big|p_0 \,.$$

The Hessian matrix at $p_1$ is given by,

$$\boxed{\mathcal{H}\Big|_{p_1} = \mathcal{H}\Big|_{p_0} + \delta\mathcal{H} \,,}$$

where we will take $\delta\mathcal{H}$ to be *stochastic*, and satisfying certain properties.

# *A novel, local approach to multifield potentials*




By repeating this procedure, the potential may be obtained to quadratic approximation along the entire path $\Gamma$.

As this method is *local,* the potential is only generated close to the path.

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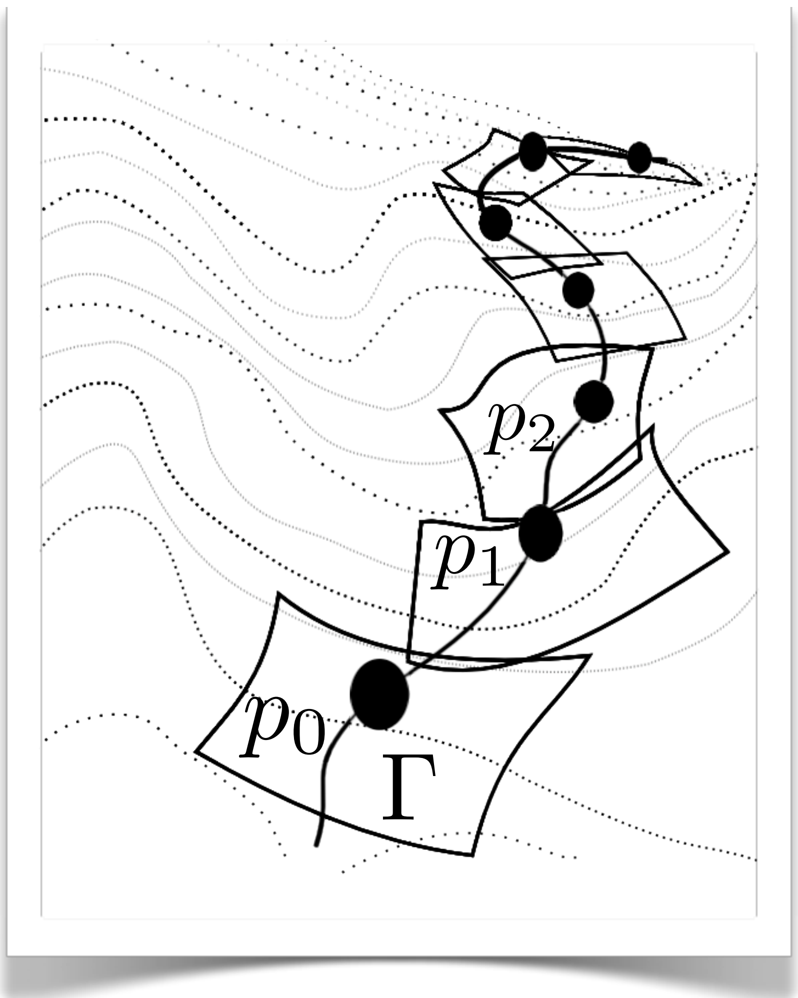


By repeating this procedure, the potential may be obtained to quadratic approximation along the entire path $\Gamma$.

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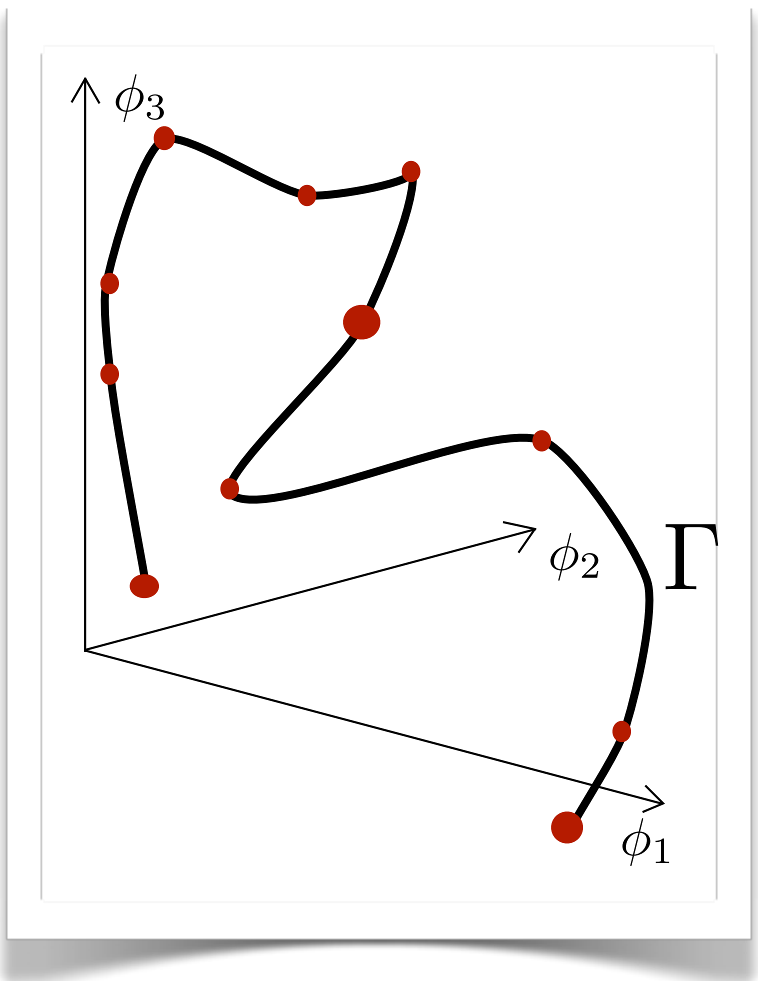
Furthermore, I have said that $\delta\mathcal{H}$ is a random matrix, but have yet to specify in what way.

# *A novel, local approach to multifield potentials*

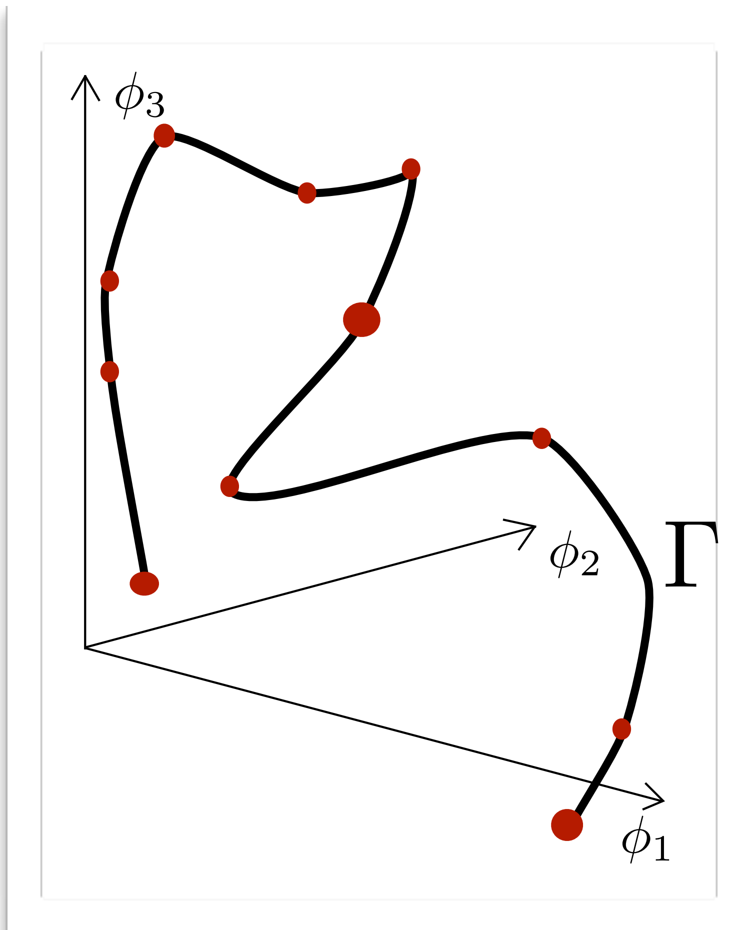



*Restrictions on $\delta\mathcal{H}$ :*

1. For a collection of well-separated points along $\Gamma$, with respect to the scale $\Lambda_h$, the corresponding collection of Hessian matrices should constitute a random sample of a rotationally invariant ensemble.

2. At each point along $\Gamma$, the $N(N+1)/2$, entries of $\mathcal{H}_{ab}$ are statistically independent.

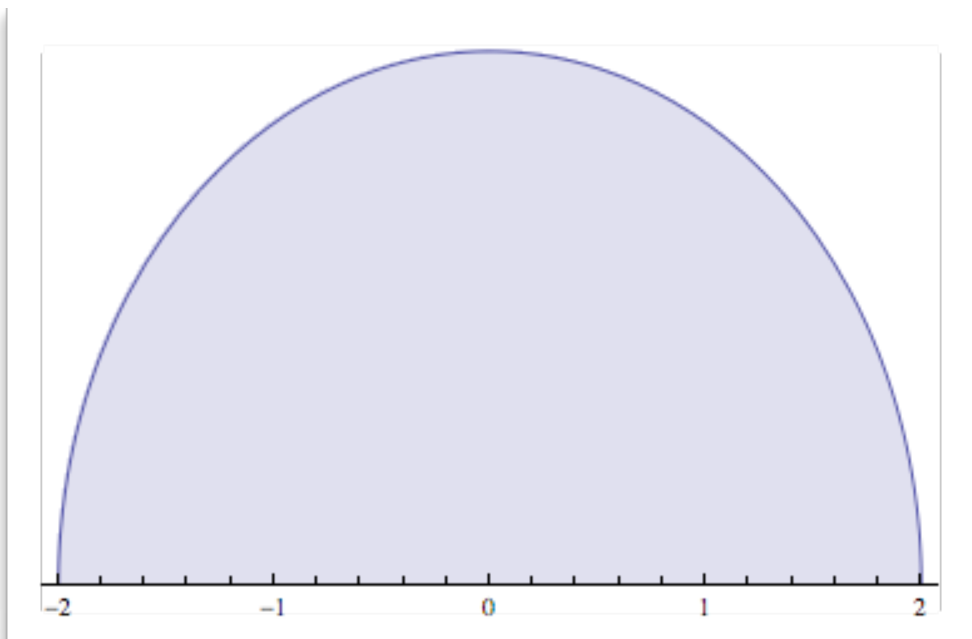
# *A novel, local approach to multifield potentials*




*Restrictions on $\delta\mathcal{H}$ :*

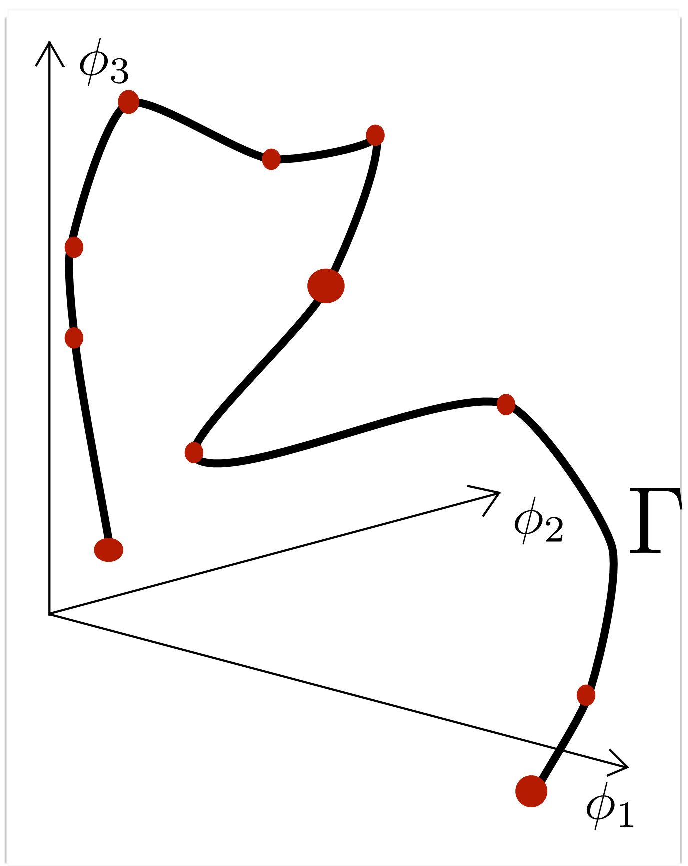
These requirements uniquely restrict the distribution of the Hessian matrix to the *Gaussian Orthogonal Ensemble (GOE)*.

At large *N*, the corresponding eigenvalue density is given by the *Wigner semi-circle*:




*See also:*
*Pedro, Westphal, 2013.*

# *A novel, local approach to multifield potentials*

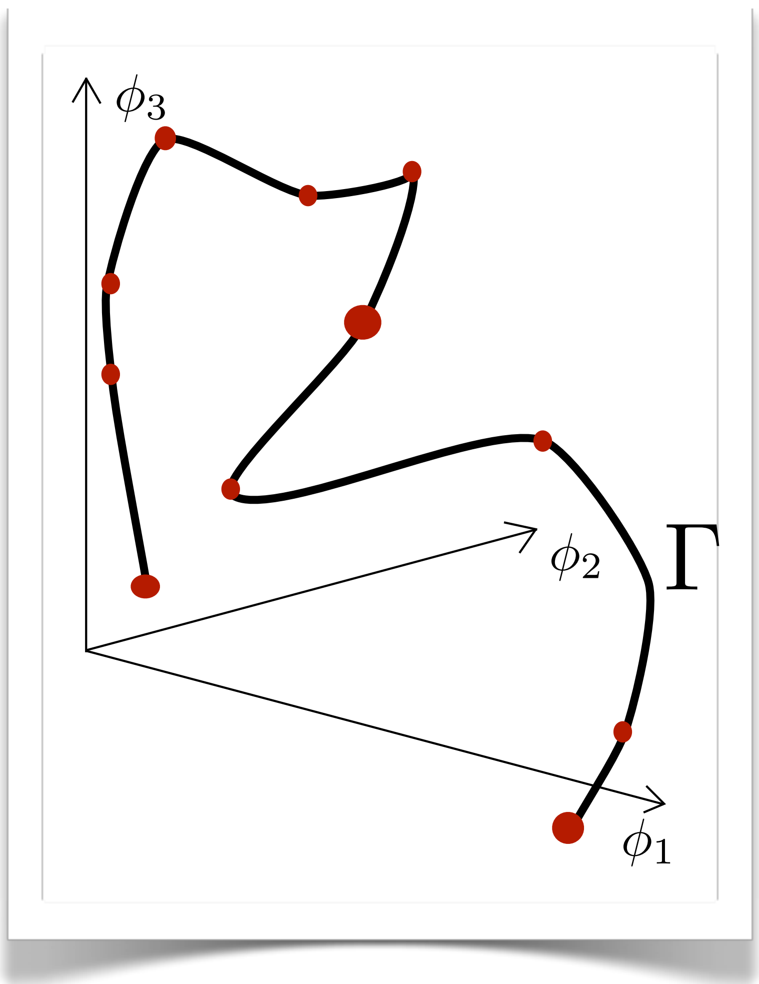



*The (stationary) GOE Coulomb gas:*

The joint probability distribution of the eigenvalues can be interpreted as a *thermal Coulomb gas* of equally charged particles in *d*=2, restricted to the real line and confined by a quadratic potential:

$$P(x_1, \ldots, x_N) \sim \exp\left[ -\frac{1}{2} \sum_a x_a^2 + \sum_{a \neq b} \ln\left(|x_a - x_b|\right) \right] .$$

# *Random Potentials from Dyson Brownian Motion*




*Thus,* the stochastic law governing the evolution of $\delta\mathcal{H}$ must be constructed so as to reproduce the GOE for well-separated points.

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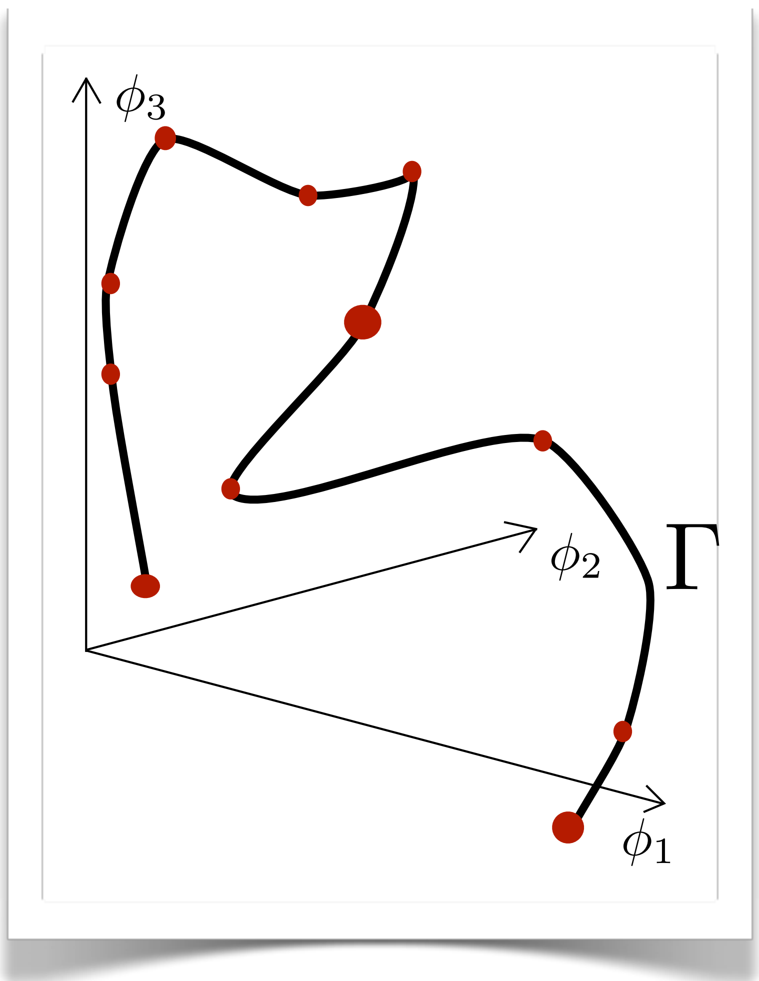


*Thus,* the stochastic law governing the evolution of $\delta\mathcal{H}$ must be constructed so as to reproduce the GOE for well-separated points.

A canonical choice for this evolution is to stipulate that the perturbation to the Hessian matrix is separable as,

$$\boxed{\delta\mathcal{H}_{ab} = \delta A_{ab} + F_{ab}(\mathcal{H})\frac{||\delta\phi^a||}{\Lambda_h}\,,}$$

where $\delta A_{ab}$ is a stochastic force, $\Lambda_h$ is the correlation length of the potential. Then, *uniquely*:

$$F_{ab}(\mathcal{H}) = -\mathcal{H}_{ab}\,.$$

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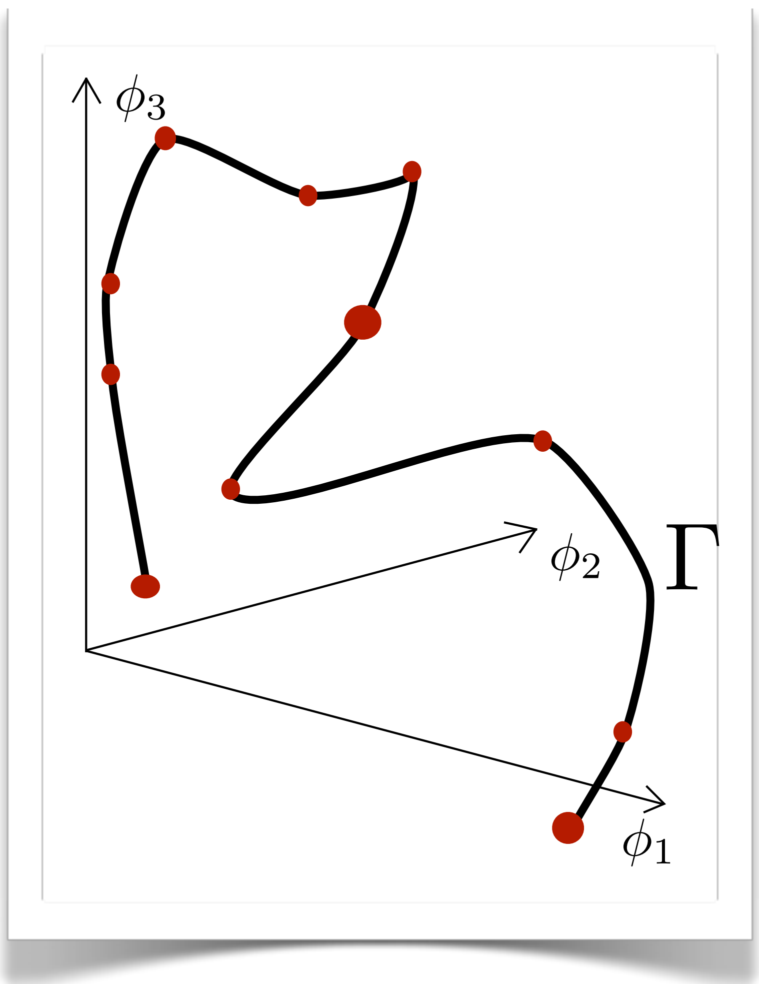


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J. Math. Phys. **3**, 140, (1962).

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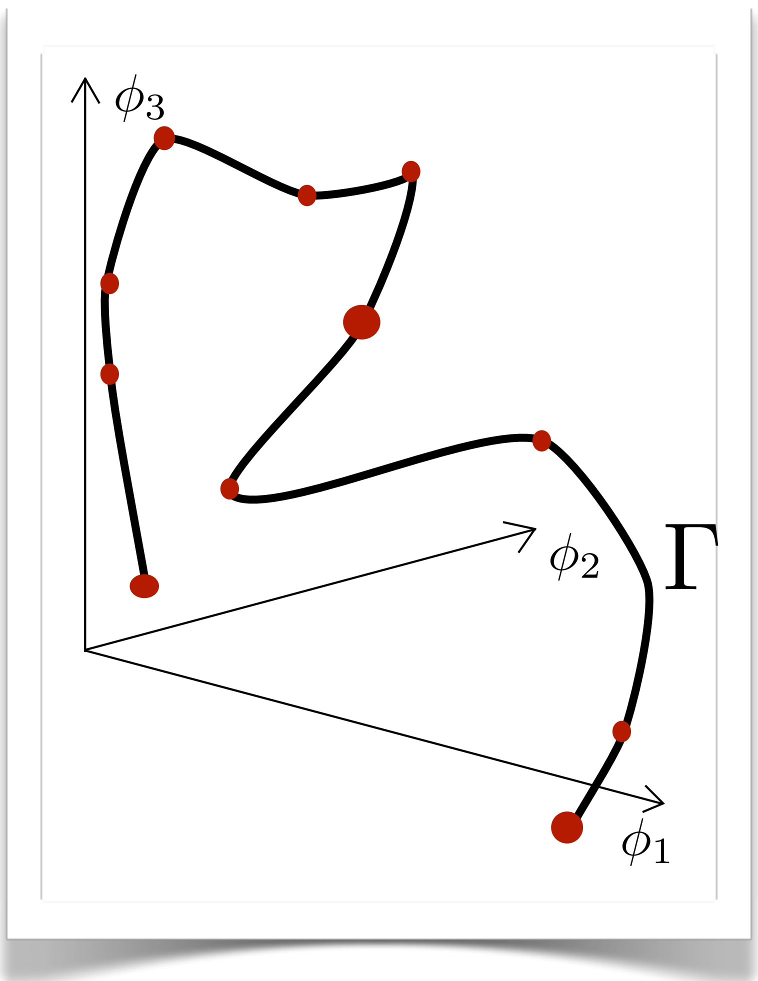


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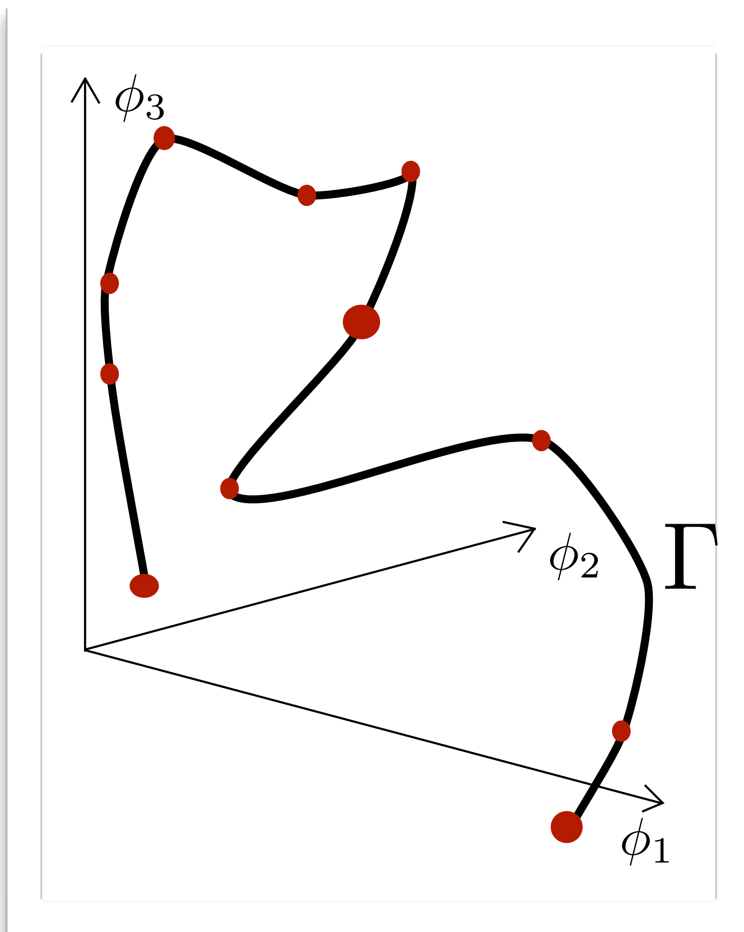
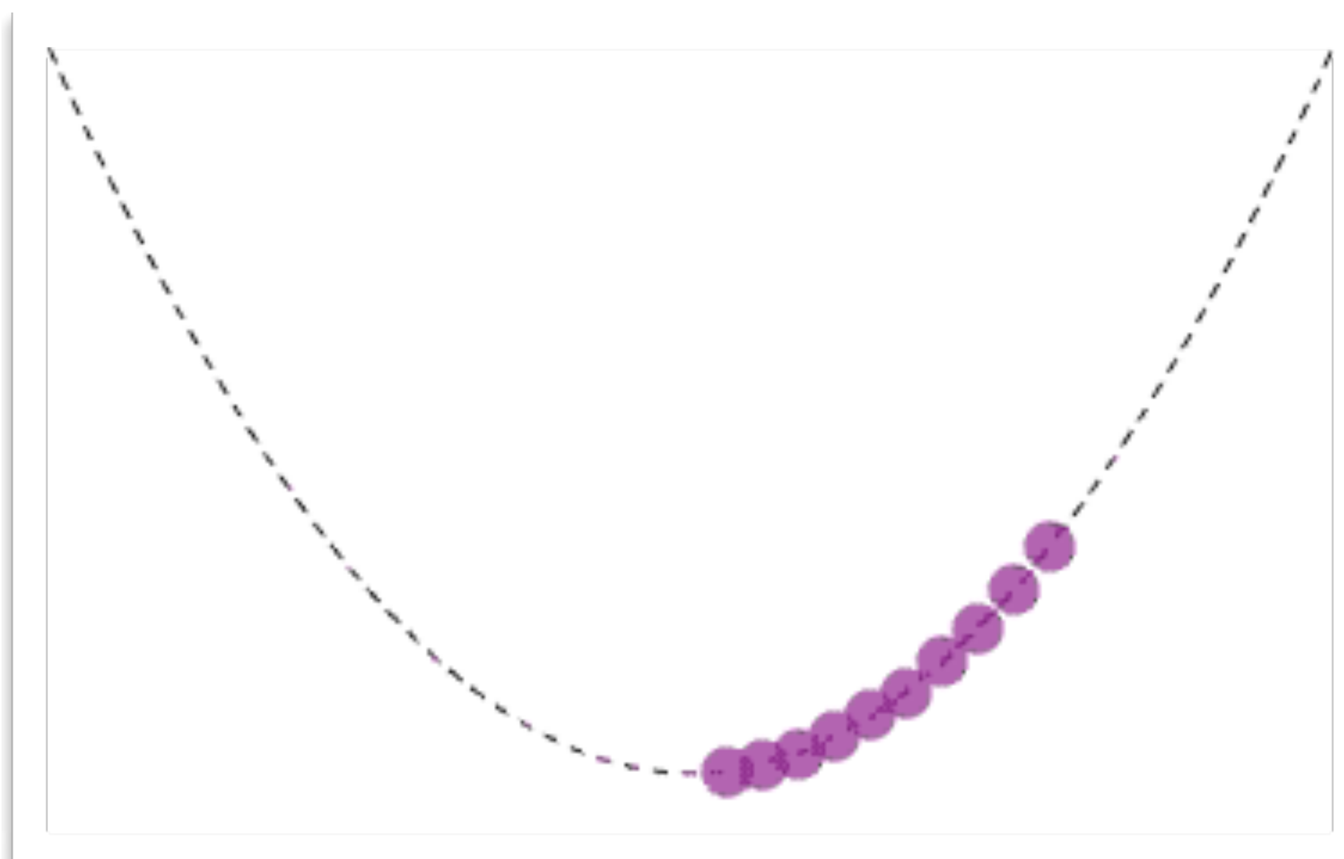



*This is Dyson Brownian Motion.*

DBM provides a non-equilibrium extension of the Coulomb gas formulation of RMT.



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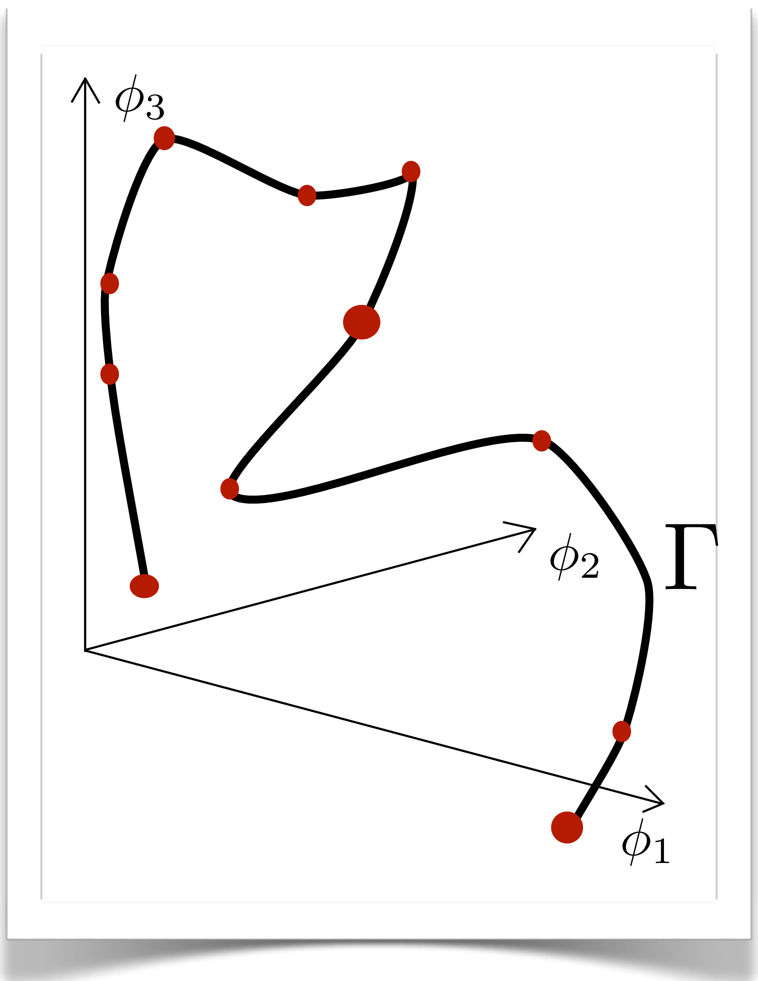


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DBM provides a non-equilibrium extension of the Coulomb gas formulation of RMT.

The probability density function of the 'time' dependent ensemble is given by,

$$P(\mathcal{H}, s) = \mathcal{C} \frac{1}{(1-q^2)^{\frac{N(N+1)}{4}}} \exp\left[-\frac{\mathrm{tr}\left(\mathcal{H}(s) - q\mathcal{H}^0\right)^2}{2\sigma^2(1-q^2)}\right],$$

where $q = \exp[-s/\Lambda_h]$ for the path length $s$ and the initial Hessian is given by $\mathcal{H}^0$.

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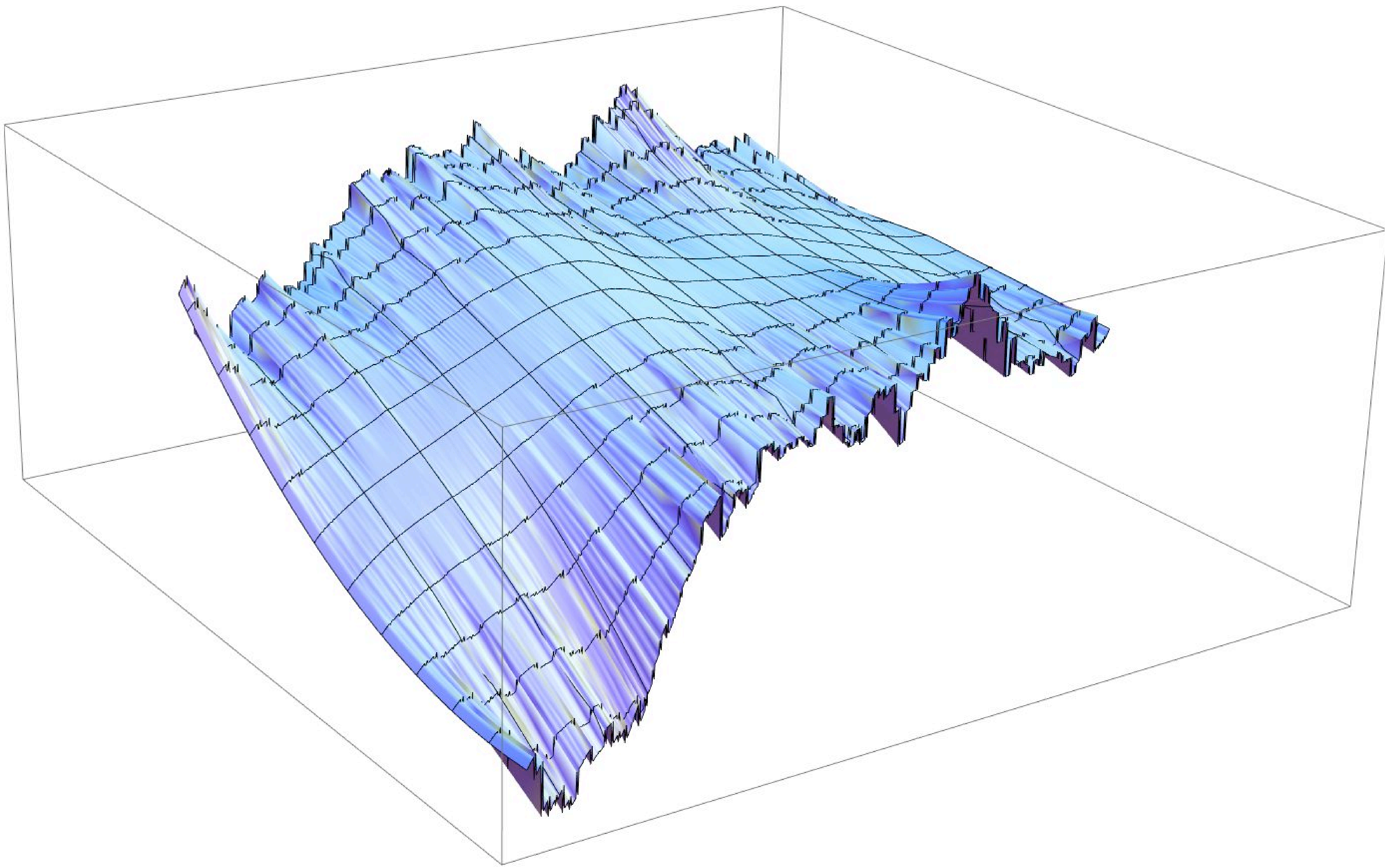

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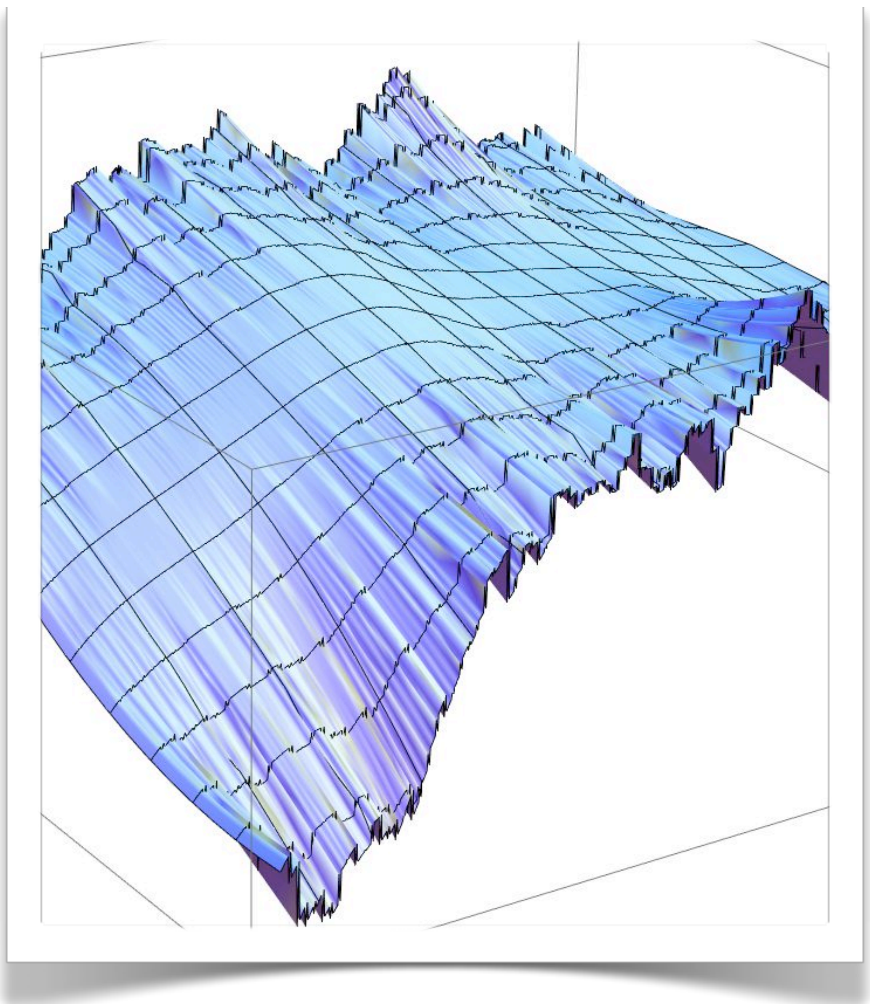


A striking feature of this new class of potentials is how efficient it is for the study of large systems.

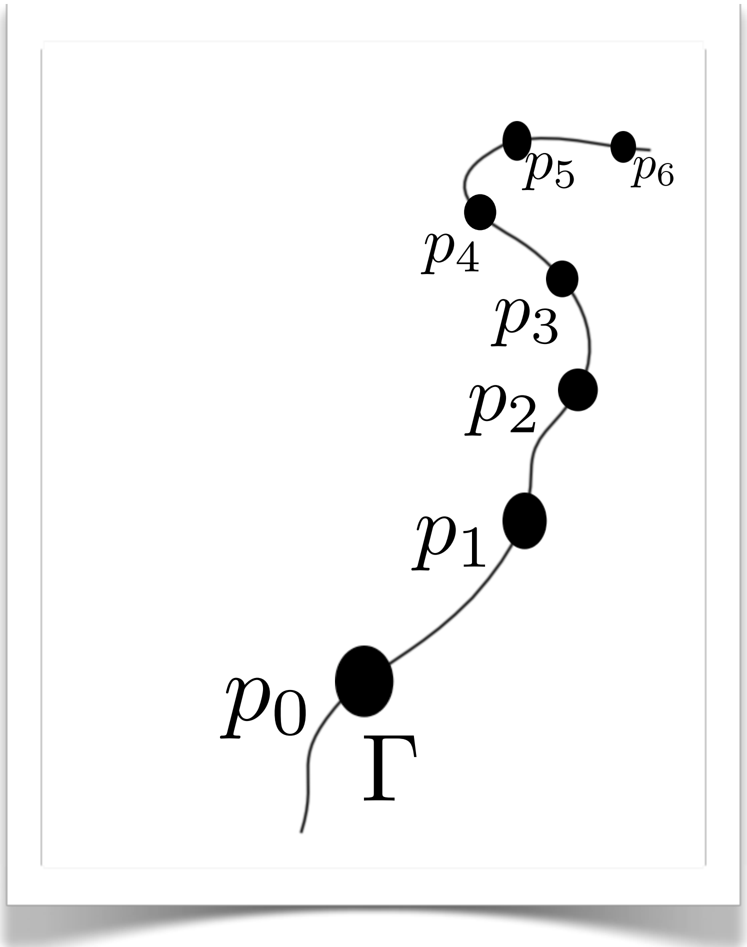
*Computational cost:*

DBM potential: $\sim N^3$ .

Random Fourier Potential: $\sim (k_{max}/k_{min})^N$ .

This allows us to study the inflationary dynamics of much larger systems than has previously been possible.

# *Charting Inflationary Landscapes with Random Matrix Theory*




A particularly interesting application for these new potentials is *multifield inflation*.

We thus take the path $\Gamma$ to be the *inflationary trajectory* in field space, and specify $\dot{\phi}^a\big|_{p_0}$ as well as the initial conditions for the potential, gradient and Hessian at $p_0$ .

The resulting inflationary dynamics is sensitive to the ratio $\Lambda_h/M_{Pl}$ , and here I will only discuss $\Lambda_h \ll M_{Pl}$ , with inflation being supported *close to an approximate critical point*.

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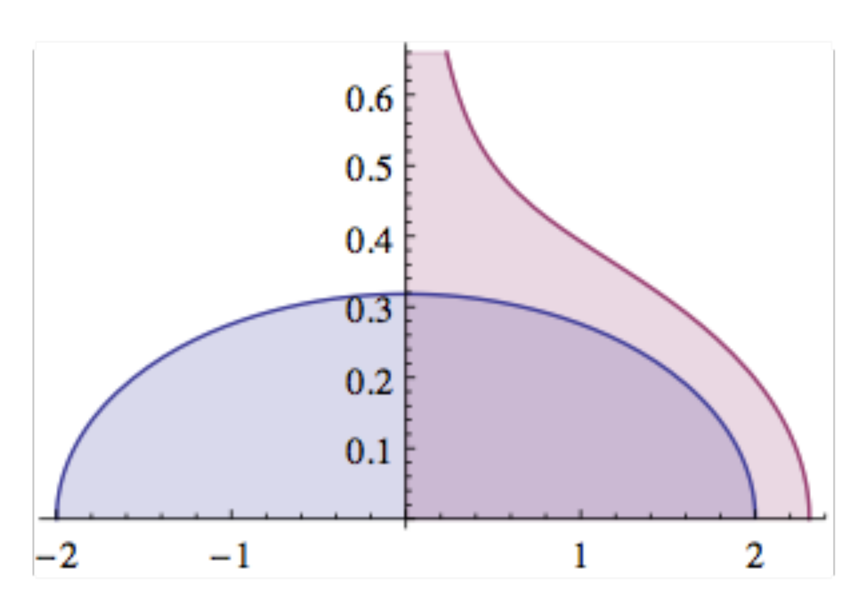


Close to an approximate critical point, sustained inflation may be supported if the gradient and the smallest eigenvalue are both small in magnitude.

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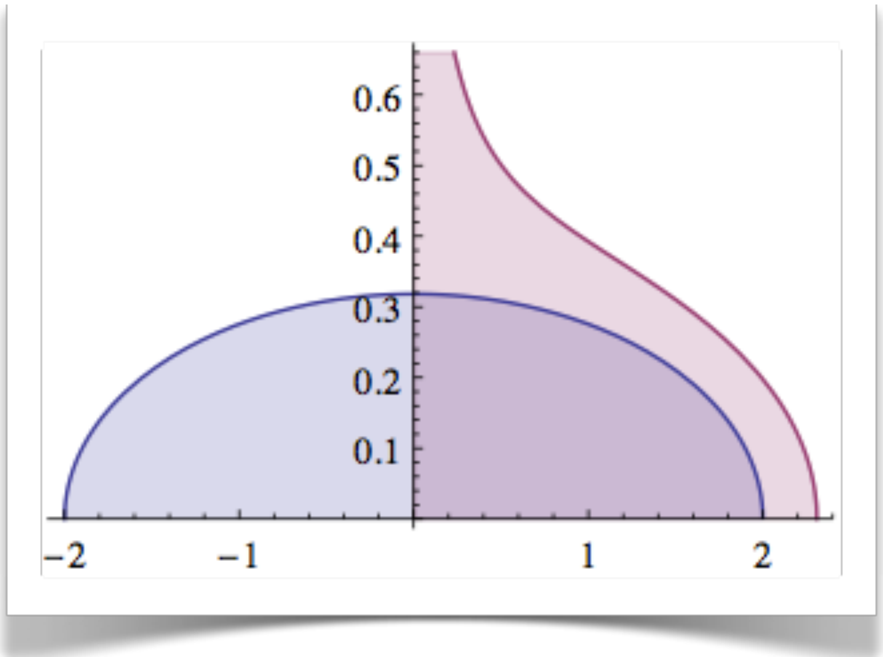


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*Characteristics for inflation with* $\Lambda_h \ll M_{pl}$ :

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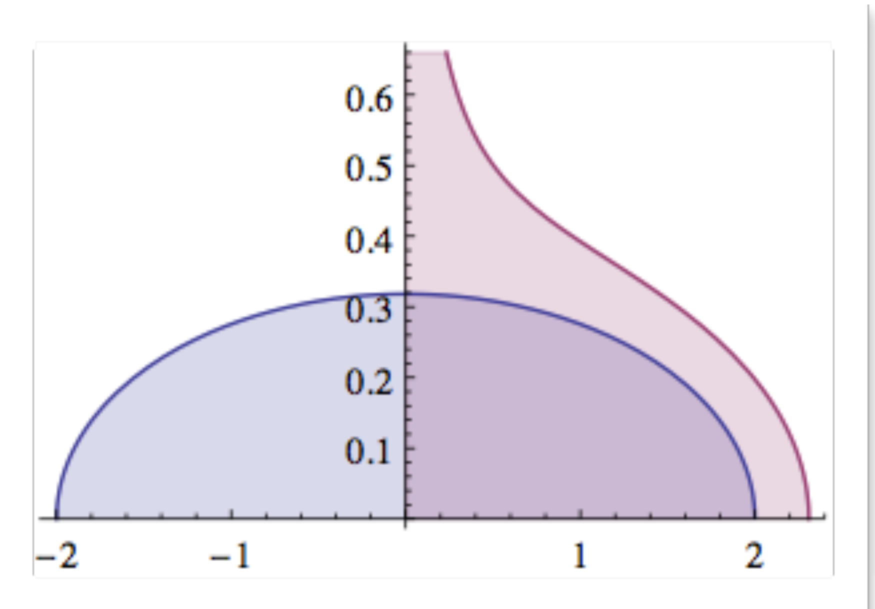


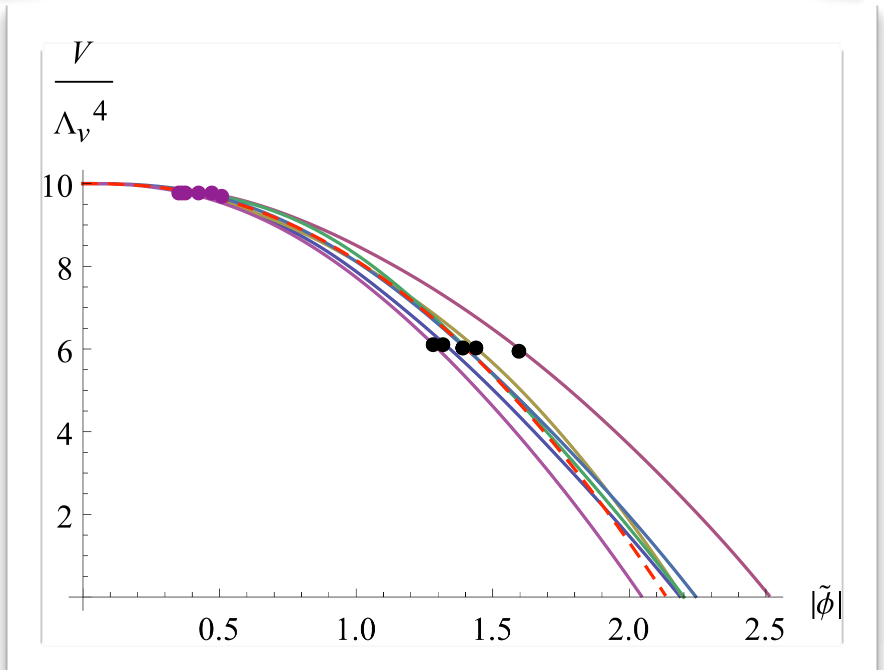


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Inflation is of *small-field* type.

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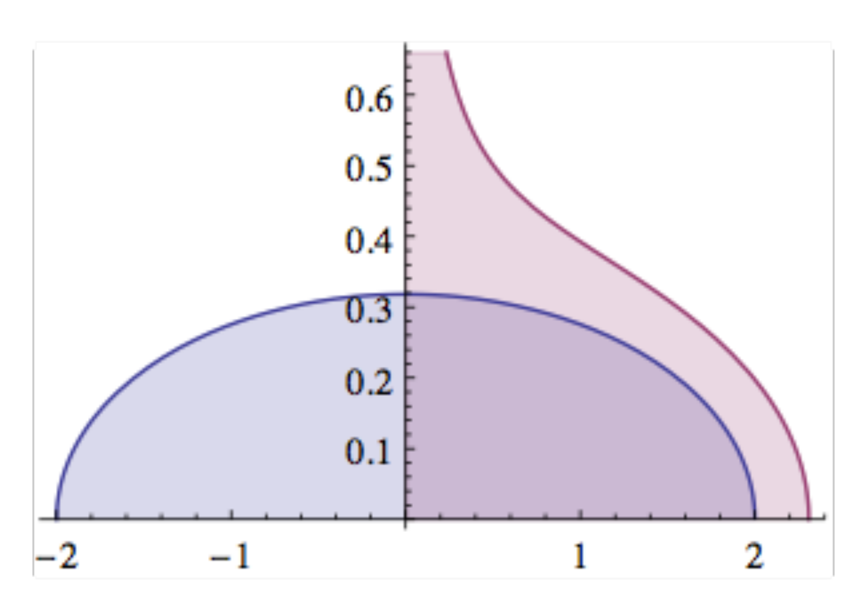


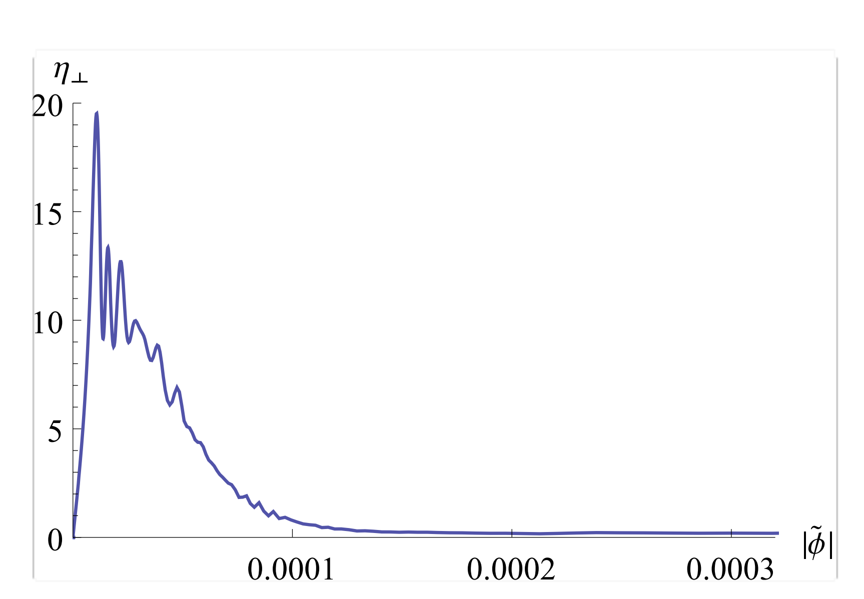


Close to an approximate critical point, sustained inflation may be supported if the gradient and the smallest eigenvalue are both small in magnitude.

*Characteristics for inflation with* $\Lambda_h \ll M_{pl}$ :

Inflation is of *small-field* type.

For random initial orientation of the gradient with respect to the eigenvectors of the Hessian, the slowly rolling *field trajectory curves* during inflation.

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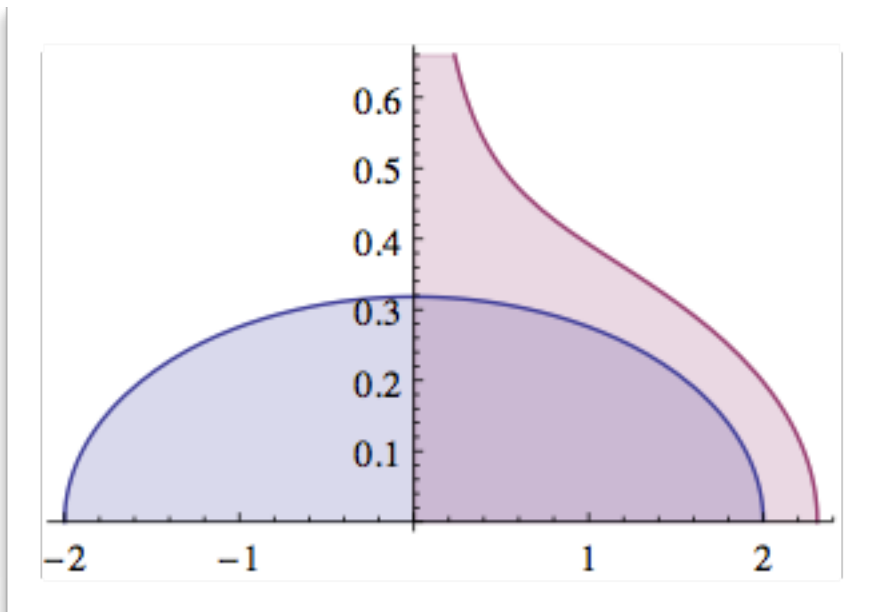


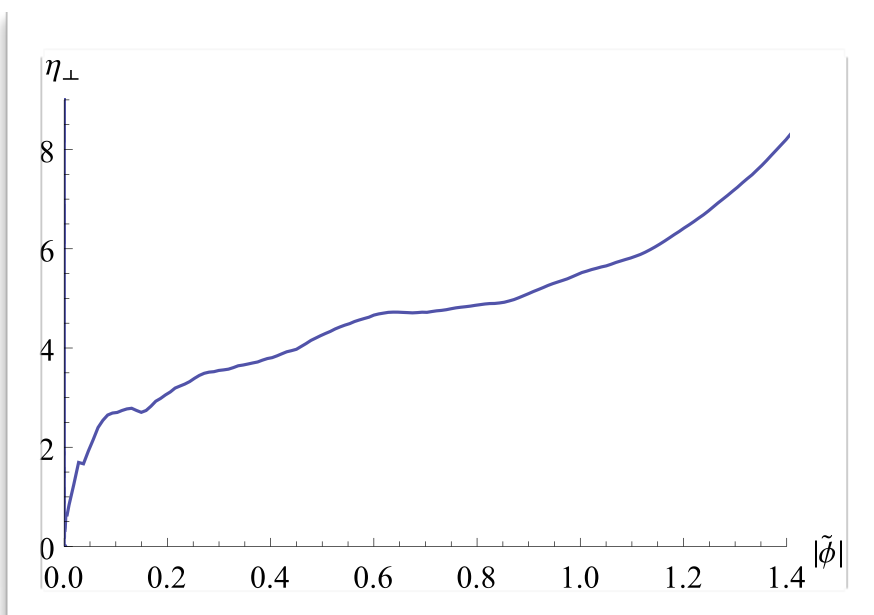


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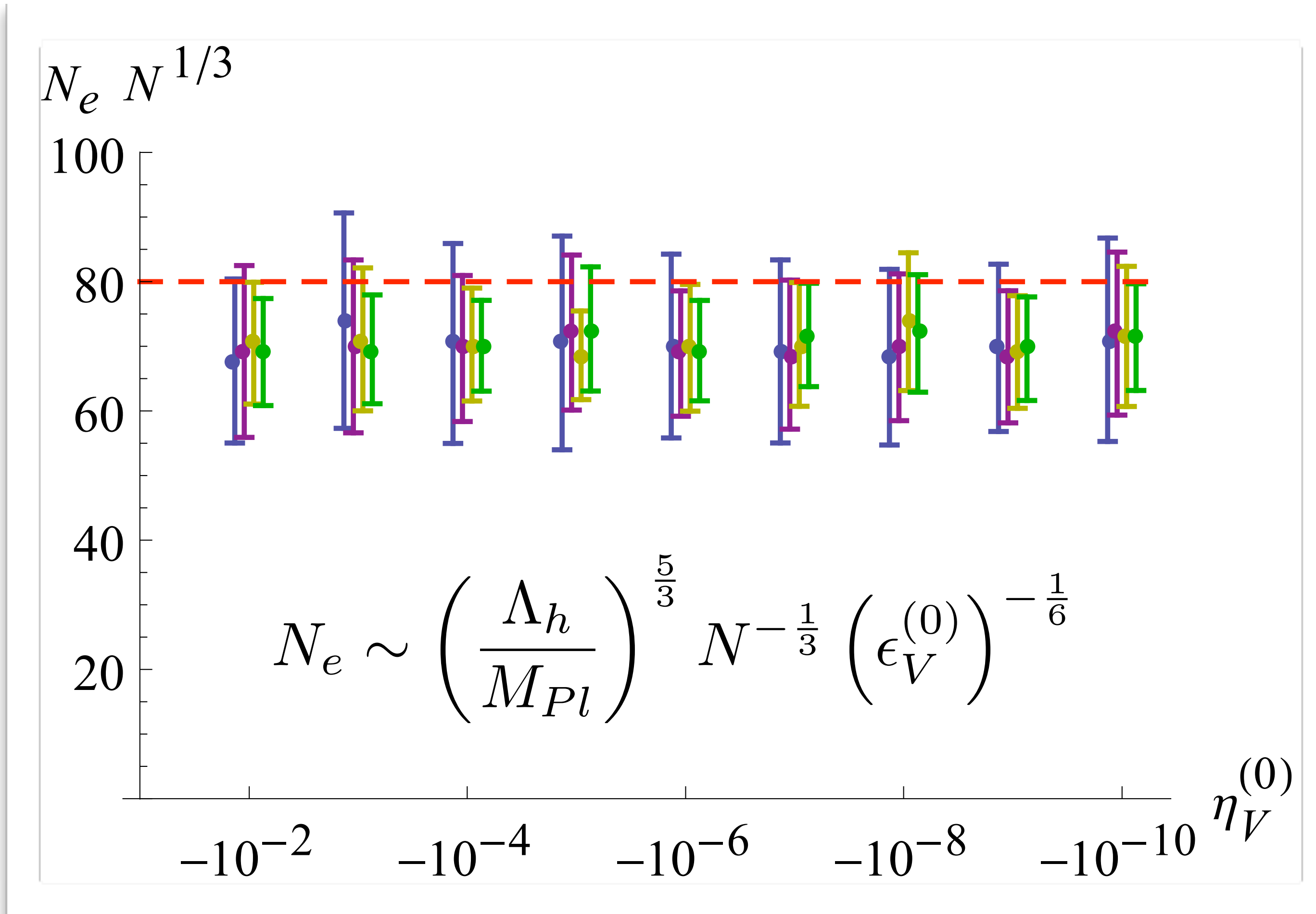
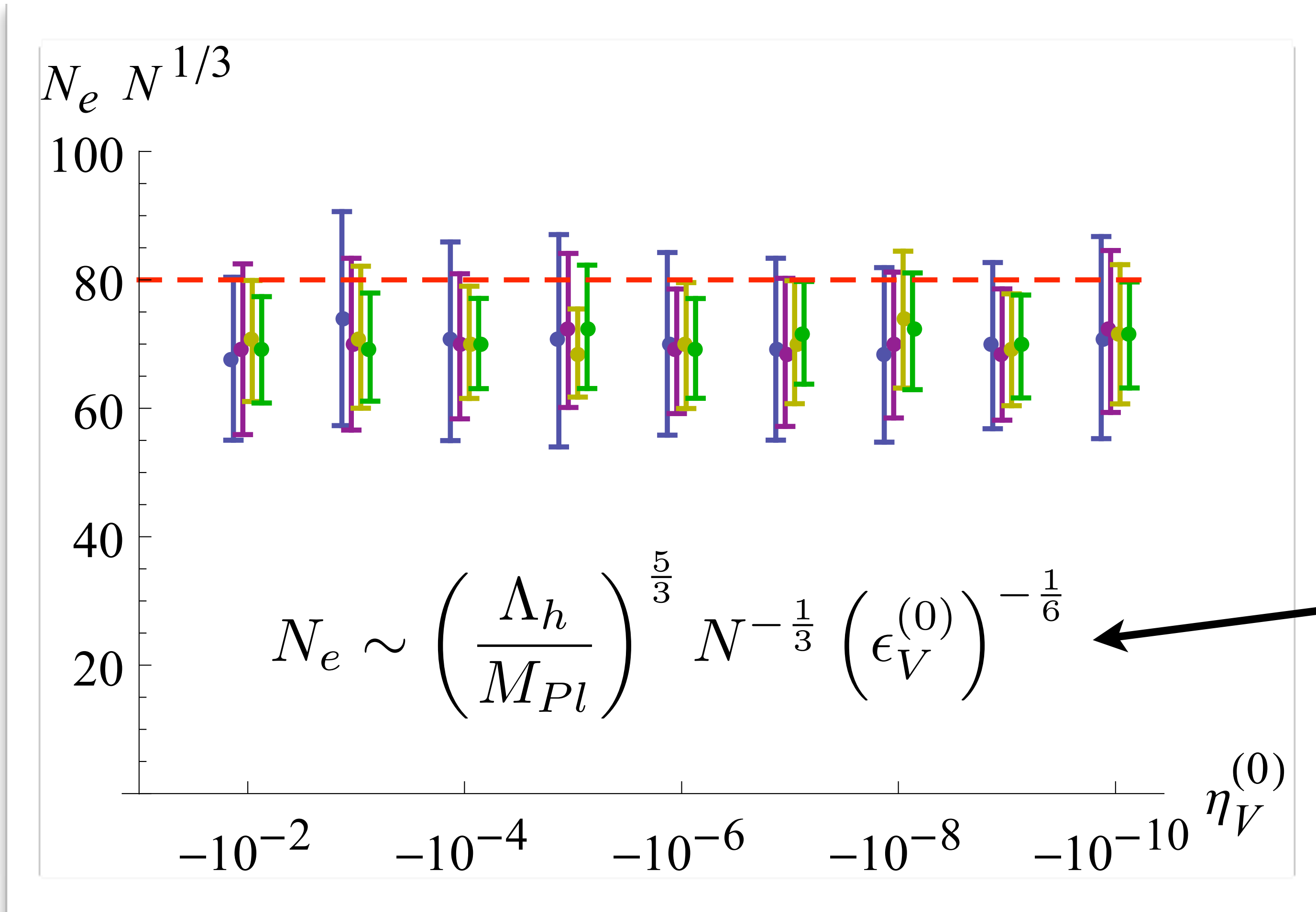



$$N = 40, 60, 80, 100, \;\; \Lambda_h/M_{Pl} = 1/10, \;\; \epsilon_V^{(0)} = 10^{-8}.$$

The number of e-folds of inflation is largely *independent* of the initial fine-tuning of $\eta_V$ .




Derived from single-field toy-model.

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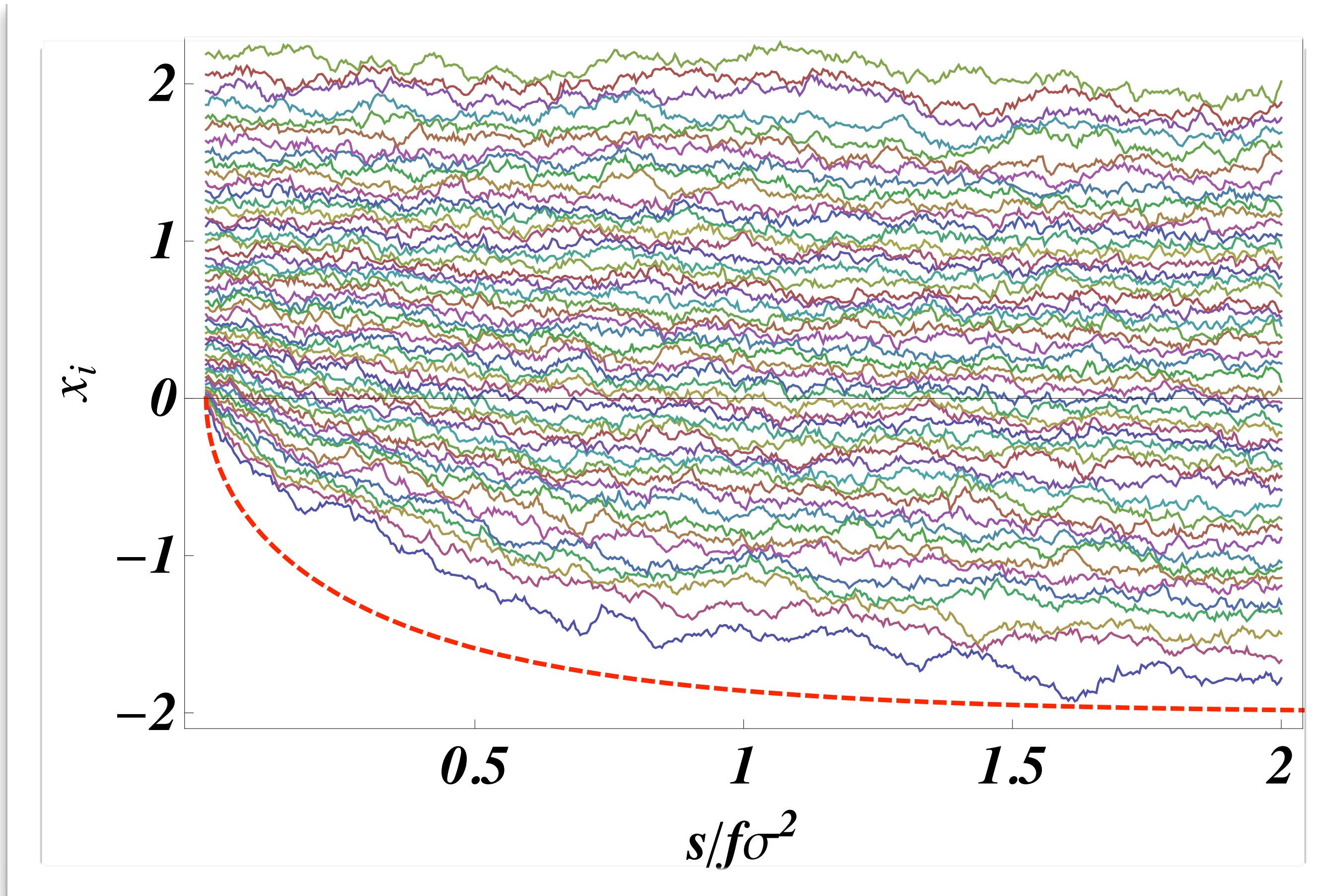


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The number of e-folds of inflation is largely *independent* of the initial fine-tuning of $\eta_V$ .

Fine-tuning of the smallest eigenvalue is quickly spoiled due to *eigenvalue relaxation*, thus reducing the number of e-folds of inflation. From RMT universality, this results can be expected to hold under quite general conditions.

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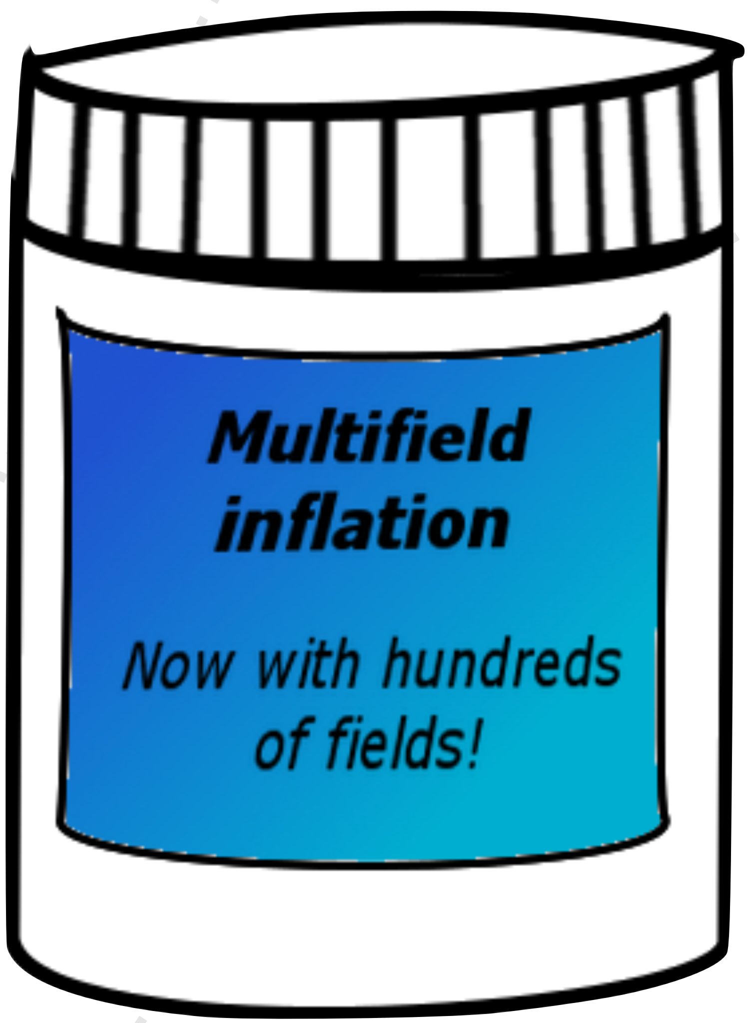


*Conclusions:*

I have presented a *novel, large class of random scalar potentials*, and a general method for constructing an even broader range of potentials.

These potentials provide an *unprecedented opportunity to study large N dynamics* of coupled scalar fields, as we have illustrated for the problem of multi-field inflation.

Multifield effects such as *eigenvalue relaxation*, can be expected to be *universal* for large classes of theories.

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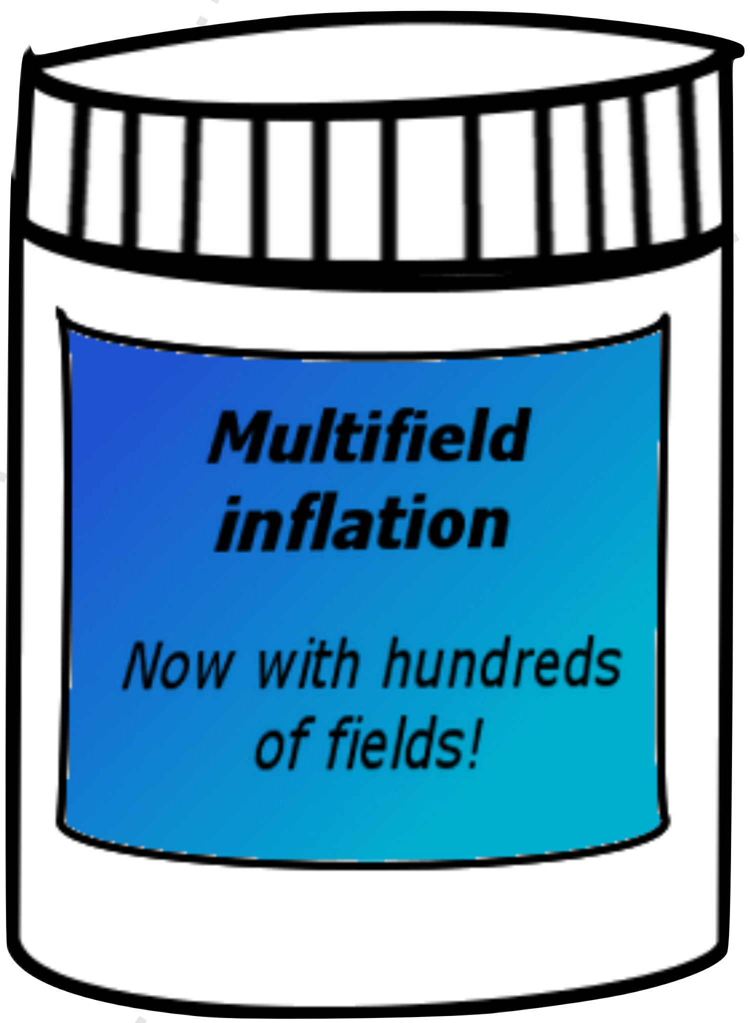


*Thanks!*


M. C. David Marsh, University of Oxford

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