# Modeling of inhomogeneity in Szekeres spacetime

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# Outline

- Metric of the Szekeres spacetime
- Geometrical interpretation of the metric functions
- Modeling of inhomogeneity

# Szekeres metric

- exact inhomogeneous dust solution that has no symmetries

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{\left(R' - R\frac{E'}{E}\right)^2}{\epsilon + f\left(r\right)}\mathrm{d}r^2 + \frac{R^2}{E^2}\left(\mathrm{d}p^2 + \mathrm{d}q^2\right) \qquad (1)$$

where $R = R(t, r)$ and

$$E\left(r, p, q\right) \equiv \frac{S\left(r\right)}{2}\left[\left(\frac{p - P\left(r\right)}{S\left(r\right)}\right)^2 + \left(\frac{q - Q\left(r\right)}{S\left(r\right)}\right)^2 + \epsilon\right] \qquad (2)$$

- $\epsilon = +1, 0, -1$
- $S(r)$, $P(r)$, $Q(r)$, $f(r)$ are arbitrary functions

but $\epsilon + f > 0$, $S \neq 0$

# Solution of dynamical equation and interpretation of $f$

$$\dot{R}^2 = \frac{2M(r)}{R} - f(r) + \frac{\Lambda R^2}{3} \tag{3}$$

- looks like Friedmann equation
- $f$ determines the curvature of the tree-spaces
  1. $f < 0$ elliptic evolution

$$R = \frac{M}{(-f)}(1 - \cos\eta), \quad \eta - \sin\eta = \frac{(-f)^{\frac{3}{2}}(t - t_B(r))}{M} \tag{4}$$

  2. $f > 0$ hyperbolic evolution

$$R = \frac{M}{(-f)}(\cosh\eta - 1), \quad \sinh\eta - \eta = \frac{f^{\frac{3}{2}}(t - t_B(r))}{M} \tag{5}$$

  3. $f = 0$ parabolic evolution

$$R = \left(\frac{9}{2}M\right)^{\frac{1}{3}}(t - t_B(r))^{\frac{2}{3}} \tag{6}$$

## $\epsilon = +1$

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{\left(R' - R\frac{E'}{E}\right)^2}{f(r) + 1}\mathrm{d}r^2 + \frac{R^2}{E^2}\left(\mathrm{d}p^2 + \mathrm{d}q^2\right) \tag{7}$$

$$E(r, p, q) \equiv \frac{S(r)}{2}\left[\left(\frac{p - P(r)}{S(r)}\right)^2 + \left(\frac{q - Q(r)}{S(r)}\right)^2 + 1\right] \tag{8}$$

Coordinate transformation

$$\frac{p - P}{S} = \cot\frac{\theta}{2}\cos\phi, \quad \frac{q - Q}{S} = \cot\frac{\theta}{2}\sin\phi \tag{9}$$

$$\frac{1}{E^2}\left(\mathrm{d}p^2 + \mathrm{d}q^2\right) = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 \tag{10}$$

$\Rightarrow$ metric on a unit sphere

# Interpretation of E and $\frac{E'}{E}$

$$g_{rr} = \frac{\left(R' - R\frac{E'}{E}\right)^2}{1+f}, \qquad 4\pi\rho = \frac{M' - 3M\frac{E'}{E}}{R^2\left(R' - R\frac{E'}{E}\right)}$$

investigate extreme values of $E$ and $\frac{E'}{E}$

$$E = \frac{S}{1 - \cos\theta} \tag{11}$$

$$E' = -\frac{S'\cos\theta + \sin\theta\,(P'\cos\phi + Q'\sin\phi)}{1 - \cos\theta} \tag{12}$$

$E' = 0$ becomes

$$S'\cos\theta + P'\sin\theta\cos\phi + Q'\sin\theta\sin\phi = 0 \tag{13}$$

$$\cos\theta = z, \quad \sin\theta\cos\phi = y, \quad \sin\theta\sin\phi = x \tag{14}$$

$$S'z + P'x + Q'y = 0 \tag{15}$$

$\Rightarrow$ equation of a plane going through the origin

$$\frac{E'}{E} = -\frac{S'\cos\theta + \sin\theta\,(P'\cos\phi + Q'\sin\phi)}{S} \tag{16}$$

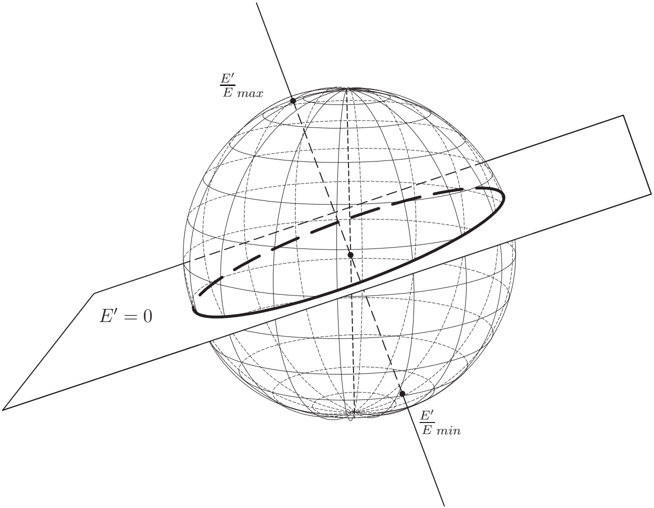
two extreme values located at $M_1 = (\theta_1, \phi_1)$ and $M_2 = (\theta_1 + \pi, \phi_1 + \pi)$

$$\cos\theta_{1,2} = \pm\frac{S'}{\sqrt{(S')^2 + (P')^2 + (Q')^2}}, \quad \cos\phi_{1,2} = \pm\frac{P'}{\sqrt{(P')^2 + (Q')^2}} \tag{17}$$

the extreme values are

$$\frac{E'}{E}_{extrem} = \pm\frac{\sqrt{(S')^2 + (P')^2 + (Q')^2}}{S} \tag{18}$$

$\Rightarrow$ a dipole


$\frac{E'}{E}_{max}$
$E' = 0$
$\frac{E'}{E}_{min}$

# Distribution of density

$$\rho = \frac{2}{\kappa c^2} \frac{M' - 3M\frac{E'}{E}}{R^2\left(R' - R\frac{E'}{E}\right)} = \frac{R'\rho_{LT} - R\frac{E'}{E}\rho_{AV}}{R' - R\frac{E'}{E}}, \tag{19}$$

where

$$\rho_{LT} \equiv \frac{2}{\kappa c^2}\frac{M'}{R^2 R'} = \frac{1}{R^2 R'}\rho_{LT0} r^2 \tag{20}$$

and

$$\rho_{AV} \equiv \frac{6}{\kappa c^2}\frac{M}{R^3} = \frac{3}{R^3}\int \rho_{LT0} r^2 \mathrm{d}r. \tag{21}$$

Behaviour of $\rho$ with respect to $\frac{E'}{E}$

$$\frac{\partial \rho}{\partial \frac{E'}{E}} = RR' \frac{\rho_{LT} - \rho_{AV}}{\left(R' - R\frac{E'}{E}\right)^2}. \tag{22}$$

$\Rightarrow$ the density also creates a dipole on each sphere

# Modeling inhomogeneity

specify on initial time frame $t_i$:

- radial density profile $\rho_{LT0}$,
- one extreme value of density $\rho_{max0}$ or $\rho_{min0}$

Compute the other extreme value of density (you can do that because of the dipole property).

Define $\Delta\rho \equiv \rho_{max0} - \rho_{min0}$ and evolve in time

## Case $f = 0$

$$\Delta\rho\left(t,r\right) = \frac{4}{3}\frac{1}{\left(t-t_B\right)^2}\frac{\Delta\rho_0}{\rho_{LT0}}h\left(t,r\right) \tag{23}$$

$$h\left(t,r\right) \equiv \frac{1-A}{A}\frac{1-\frac{R'}{R}rA}{\left(\frac{R'}{R}r\right)^2\left(D-A\right)\left(C-A\right)+\left(D-1\right)\left(C-1\right)} \tag{24}$$

$$A \equiv \frac{3\int \rho_{LT0}r^2\mathrm{d}r}{\rho_{LT0}r^3}, \quad D \equiv \frac{\rho_{max0}}{\rho_{LT0}}, \quad C \equiv \frac{\rho_{min0}}{\rho_{LT0}} \tag{25}$$

$$\frac{R'}{R}r = \frac{1}{A} + \frac{2}{3}\frac{A-1}{A^{\frac{3}{2}}}\frac{1}{t-t_B}\sqrt{\frac{3}{\rho_{LT0}}}. \tag{26}$$

# Behaviour of $h$ for small initial inhomogeneity

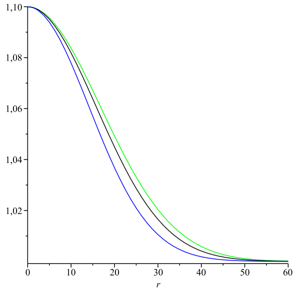
$$h(t,r) \approx \frac{2\sqrt{3}}{3}\frac{1}{t-t_B}\frac{1}{\rho_{LT0}^{\frac{3}{2}}} \tag{27}$$

$$\Delta\rho(t,r) \approx \frac{8\sqrt{3}}{9}\frac{1}{(t-t_B)^3}\frac{\Delta\rho_0}{\rho_{LT0}^{\frac{3}{2}}}. \tag{28}$$

we choose

$$\rho_{LT0} = 1 + \frac{1}{10}e^{-\frac{r^2}{500}}, \quad \rho_{min0} = 1 + \frac{1}{10}e^{-\frac{r^2}{400}} \tag{29}$$

$$t_i = 5 \cdot 10^5 y \quad t_f = 14 \cdot 10^9 y \tag{30}$$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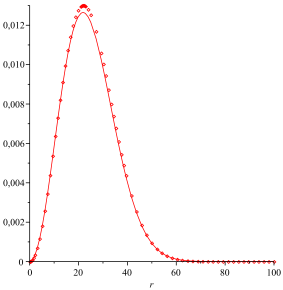
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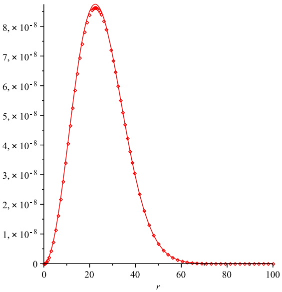
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