Trans-Planckian Considerations in Inflationary Cosmology

Or: Can unknown high energy physics alter predictions from inflationary cosmology?

Benedict J. Broy

II. Institut für Theoretische Physik,
Universität Hamburg

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Recall the standard picture

\[ \hat{\phi}(\tau, x) = \bar{\phi}(\tau) + \delta \hat{\phi}(\tau, x) \]

Fluctuation

Introducing a rescaled field \( \hat{f} = a(\tau) \hat{\phi} \), it may be shown that

\[ f_k'' + \left( k^2 - \frac{z''}{z} \right) f_k = 0, \]

where \( z = a \bar{\phi}' / H \) with \( H = a \dot{H} \). For slow-roll in de Sitter we have

\[ \frac{z''}{z} \approx \frac{a''}{a} = \frac{2}{\tau^2}. \]
Quantum Fluctuations in de Sitter Space II

Hence, recast above as

\[ f''_k + \left( k^2 - \frac{2}{\tau^2} \right) f_k = 0. \]

We need to define a vacuum \( a_k |0\rangle = 0 \). Idea: Go to the infinite past and to arbitrarily short distances to let space resemble Minkowski space, i.e.

\[ f''_k + k^2 f_k = 0. \]

Thus have the solution

\[ f_k(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau}. \]
The general solution is

\[ f_k(\tau) = A_k \frac{e^{-ik\tau}}{\sqrt{2k}} \left( 1 - \frac{i}{k\tau} \right) + B_k \frac{e^{ik\tau}}{\sqrt{2k}} \left( 1 + \frac{i}{k\tau} \right) \]

Combine initial condition and general solution to yield unique mode function

\[ f_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \left( 1 - \frac{i}{k\tau} \right). \]
The vacuum fluctuations of the field $\delta \hat{\phi} = a^{-1}\hat{f}$ may be calculated as

$$\langle 0 | \hat{\phi}_k^\dagger \hat{\phi}_{k'} | 0 \rangle = \delta(k - k')a^{-2}|f_k|^2$$

$$= \frac{H^2}{2k^3}(1 + k^2\tau^2)$$

$$\rightarrow \frac{H^2}{2k^3},$$

or

$$P_\phi = \frac{k^3}{2\pi^2} \langle 0 | \hat{\phi}_k^\dagger \hat{\phi}_{k'} | 0 \rangle = \left( \frac{H}{2\pi} \right)^2.$$
Recall the standard setting

\[ f_k'' + \left( k^2 - \frac{z''}{z} \right) f_k = 0. \]

Replace \( k \rightarrow aF(k, \tau) \) where \( aF \rightarrow k \) for \( k \ll k_c \) for some cutoff scale \( k_c \). We thus have

\[ f_k''' + \left( a^2 F^2 - \frac{z''}{z} \right) f_k = 0. \]
Consider WKB solution

\[ f_k^{WKB}(\tau) = \frac{1}{\sqrt{2\omega(\tau)}} e^{-i \int_{\tau_i}^{\tau} \omega' d\tau'}, \]

where the exact solution may be cast as

\[ f_k(\tau) = A_k(\tau) f^{WKB}(\tau) + B_k(\tau) (f^{WKB}(\tau))^*. \]

For our initial condition, \( B_k(\tau) = 0 \) unless \( \omega \) violates adiabaticity, i.e. \( \partial_\tau \omega / \omega^2 = O(1) \). A violation yields

\[ P_\phi \propto k^3 |f_k|^2 \]

\[ = \left| f_k^{(0)} \right|^2 \underbrace{[|A_k|^2 + |B_k|^2 + A_k^* B_k + B_k^* A_k]}_{B_k \to 0}. \]
Approach 2: New Physics Hypersurface I

Idea: Introduce timelike hypersurface, i.e. physical cut-off scale $\Lambda$, where mode evolution starts. As comoving and physical momentum are related via $k = a p$, mode evolution starts once $k = a \Lambda$. In de Sitter, $a = -(H \tau)^{-1}$, thus have initial time of mode evolution

$$\tau_i = -\frac{\Lambda}{H k}.$$ 

Influence of unknown physics is then encoded in initial conditions. Consider

$$f_k(\tau) = A_k \frac{e^{-ik\tau}}{\sqrt{2k}} \left( 1 - \frac{i}{k \tau} \right) + B_k \frac{e^{ik\tau}}{\sqrt{2k}} \left( 1 + \frac{i}{k \tau} \right),$$
and find its conjugate momentum to be
\[
g_k(\tau) = A_k \sqrt{\frac{k}{2}} e^{-ik\tau} - B_k \sqrt{\frac{k}{2}} e^{ik\tau}.
\]

When fixing the vacuum
\[
a_k(\tau_i)|0, \tau_i\rangle = 0
\]
and considering \( |A_k|^2 - |B_k|^2 = 1 \), one then finds
\[
B_k = \frac{ie^{-2ik\tau_i}}{2k\tau_i + i} A_k.
\]
Approach 2: New Physics Hypersurface III

Computing the power spectrum yields

\[
P_\phi = \frac{k^3}{2\pi^2 a^2} \langle 0 | f_k^\dagger f_{k'} | 0 \rangle
\]

\[
\rightarrow \left( \frac{H}{2\pi} \right)^2 \left( |\mathcal{A}_k|^2 + |\mathcal{B}_k|^2 + \mathcal{A}_k^* \mathcal{B}_k + \mathcal{A}_k \mathcal{B}_k^* \right),
\]

\[
B_k = 0
\]

From the above, one can deduce

\[
|\mathcal{A}_k|^2 = \left( 1 - \left| \frac{i}{2k\tau + i} \right|^2 \right)^{-1}
\]
and

$$|B_k|^2 = \frac{1}{(2k\tau_i)^2} = \frac{H^2}{4\Lambda^2}$$

so that

$$P_\phi = \left(\frac{H}{2\pi}\right)^2 \left[ 1 + |B_k|^2 \left( 2 + \frac{2k\tau_i + i}{i} e^{2ik\tau_i} + \frac{2k\tau_i - i}{-i} e^{-2ik\tau_i} \right) \right]$$

$$= \left(\frac{H}{2\pi}\right)^2 \left[ 1 + \mathcal{O}\left(\frac{H}{\Lambda}\right) \right].$$
Having considered an initial time

\[ \tau_i = -\frac{\Lambda}{Hk} \]

above might suggest to use the formalism and choose \( \tau_i \) to be the time of the onset of inflation. As spacetime might not be de Sitter before \( \tau_i \), access to the infinite past is lost and Bunch-Davies prescription is inconsistent? However above approach fails as onset of inflation \( \neq \) physical cut-off! So more work required...
When considering a non-Bunch-Davies state, additional energy density $\langle \rho \rangle = \langle 0 | T^0_0 | 0 \rangle$ is present, which must not be bigger than that of the background. One finds

$$\langle \rho(\tau) \rangle = \frac{1}{4\pi^2} \int_H^C dp |B_k|^2 \left( p^3 + \frac{1}{2} H^2 p \right).$$

For a physical cut-off $C = \Lambda$, one thus requires

$$|B_k|^2 < \frac{M_{pl}^2 H^2}{\Lambda^4}.$$
However, a physical cut-off requires a mode creation mechanism for the finite excitations not to inflate away. Compare with comoving cut-off $C = a^{-1} \Lambda$ so that

$$\langle \rho(\tau) \rangle \propto B_k \frac{\Lambda}{a^4},$$

which inflates away exponentially.
Due to a reparametrisation, this initial state is called $\alpha$-vacuum

\[
A_k = \frac{1}{\sqrt{1 - e^{\alpha + \alpha^*}}} \\
B_k = \frac{e^\alpha}{\sqrt{1 - e^{\alpha + \alpha^*}}},
\]

where $\alpha \to -\infty$ corresponds to the Bunch-Davies state.
Let us couple a detector to the inflaton field

\[ i c \langle E_1, \phi_1 | \int m(\tau) \phi[x(\tau)] d\tau | 0, E_0 \rangle \]

For the Bunch-Davies vacuum one finds

\[ R_{ij} \rightarrow \frac{\rho(E_j)}{\rho(E_i)} = e^{-2\pi \delta E}, \]

yet for an \( \alpha \)-vacuum one has

\[ R_{ij} \rightarrow e^{2\alpha}, \]

i.e. it is non-thermal!
The below expression is divergent

\[ \langle \rho \rangle = \frac{1}{2(2\pi)^3 a^4} \int d^3 k \cdot k \left[ (|A_k|^2 + |B_k|^2) \left( 1 + \frac{1}{2(k\tau)^2} \right) \right], \]

regularising the \( \alpha \)-contribution does not regularise the Bunch-Davies part, whereas for the usual Bunch-Davies vacuum, any regulator regulates flat space as well.
Consider e.g. the following action (arXiv:hep-th/0201158)

\[ S_{\text{eff}}[\phi] = \int d^4p \phi(p)\phi(-p)\left[ \frac{p^2}{2} + \frac{H^2}{2} + c_0 H^2 \left( \frac{H^2}{\Lambda^2} \right) + c_1 p^2 \left( \frac{H^2}{\Lambda^2} \right) + c_2 \left( \frac{p^4}{\Lambda^2} \right) + c_3 \left( \frac{p^4}{\Lambda^2} \right) \left( \frac{H^2}{\Lambda^2} \right) + c_4 \frac{p^6}{\Lambda^4} + \ldots \right]. \]

which then leads to

\[ \langle \phi(p)\phi(-p) \rangle |_{p=H} = H^2 (1 + c_0 \left( \frac{H^2}{\Lambda^2} \right) + c_1 \left( \frac{H^2}{\Lambda^2} \right) + c_2 \left( \frac{H^2}{\Lambda^2} \right) + c_3 \left( \frac{H^2}{\Lambda^2} \right)^2 + c_4 \left( \frac{H^2}{\Lambda^2} \right)^2 + \ldots). \]
Conclusions

- Modifying Dispersion relation shows that in principle modifications of the power spectrum are possible, yet specific form hard to motivate
- Introducing a physical cut-off scale looks promising at first but brings with it severe challenges
- In the end, EFT based approaches give corrections quadratic in $H/\Lambda$
- Bunch-Davies prescription valid even if spacetime might not be de Sitter before some finite $\tau_i$?

Thank you very much for your attention!