Trying to understand the instability of AdS through toy models

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Outline

- Brief reminder on the conjectured instability of AdS
- Some questions this conjecture has raised
- *"...in most cases where we seek in vain the answer to a question, the cause of the failure lies in the fact that problems simpler and easier than the one in hand have been either not at all or incompletely solved. All depends, then, on finding out these easier problems, and on solving them by means of devices as perfect as possible and of concepts capable of generalization."* (David Hilbert)
- Toy models: nonlinear waves on compact manifolds
  - Cubic wave equation on a torus
  - Yang-Mills equation on the Einstein universe
  - Wave map equation on the Einstein universe
- Conclusions

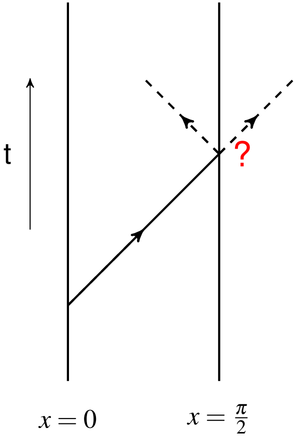
Anti-de Sitter spacetime in $d+1$ dimensions

Manifold $\mathscr{M} = \{t \in \mathbb{R}, x \in [0, \pi/2), \omega \in S^{d-1}\}$ with metric

$$g = \frac{\ell^2}{\cos^2 x}\left(-dt^2 + dx^2 + \sin^2 x \, d\omega^2_{S^{d-1}}\right)$$

Spatial infinity $x = \pi/2$ is the timelike cylinder $\mathscr{I} = \mathbb{R} \times S^{d-1}$ with the boundary metric $ds^2_{\mathscr{I}} = -dt^2 + d\Omega^2_{S^{d-1}}$

- Null geodesics get to infinity in finite time (but infinite affine length)
- AdS is **not globally hyperbolic** - to make sense of evolution one needs to choose boundary conditions at $\mathscr{I}$
- Asymptotically AdS spacetimes by definition have the same conformal boundary as AdS



Is AdS stable?

- By the positive energy theorem AdS space is the unique ground state among asymptotically AdS spacetimes (much as Minkowski space is the unique ground state among asymptotically flat spacetimes)
- Basic question for any equilibrium solution: **do small perturbations of it at $t = 0$ remain small for all future times?**
- Minkowski spacetime was proved to be asymptotically stable by Christodoulou and Klainerman (1993)
- The question of stability of AdS has not been explored until recently (notable exceptions: Friedrich 1995, Anderson 2006, Dafermos 2006)
- Key difference between Minkowski and AdS: **the main mechanism of stability of Minkowski - dissipation of energy by dispersion - is absent in AdS** (for no flux boundary conditions $\mathscr{I}$ acts as a mirror)

AdS gravity with a spherically symmetric scalar field

Conjecture (B-Rostworowski 2011, Jałmużna-Rostworowski-B 2011)

AdS$_{d+1}$ (for $d \geq 3$) is unstable against the formation of a black hole for a large class of arbitrarily small perturbations

Evidence:

- **Perturbative:** resonant interactions between harmonics give rise to secular terms at higher orders of the formal perturbation expansion. This **shifts the energy spectrum to higher frequencies**. The same happens for vacuum Einstein equations (Dias-Horowitz-Santos 2011).
- **Heuristic:** the transfer of energy to higher frequencies (or equivalently, concentration of energy on finer and finer spatial scales) is expected to be eventually cut off by horizon formation.
- **Numerical:** perturbation analysis breaks down when $\varepsilon \sim t\varepsilon^3 \Rightarrow$ **perturbations of size ε start growing after time $\mathcal{O}(\varepsilon^{-2})$.** Subsequent nonlinear evolution leads to the black hole formation (confirmed independently by Buchel-Lehner-Liebling 2012).

Follow-up studies and questions

- **Turbulent instability is absent for some initial data**: one-mode data (B-Rostworowski 2011), fat gaussians (Buchel-Lehner-Liebling 2013), time-periodic solutions in vacuum (Dias-Horowitz-Santos 2011) and for the Einstein-scalar (Maliborski-Rostworowski 2013), standing waves (Buchel-Liebling-Lehner 2013, Maliborski-Rostworowski 2014).
- How big are these stability islands on the turbulent ocean?
- Is the fully resonant linear spectrum necessary for the turbulent instability? (Dias, Horowitz, Marolf, Santos 2012). Is it sufficient?
- Energy cascade has the power-law spectrum $E_k \sim k^{-\alpha}$ with a universal exponent α (B-Rostworowski 2012). What determines α?
- Weakly turbulent instability of AdS$_3$: small smooth perturbations of AdS$_3$ remain smooth for all times but their radius of analyticity shrinks to zero as $t \to \infty$ (B-Jałmużna 2013)
- **What happens outside spherical symmetry?** It is not clear at all if the natural candidate for the endstate of instability - Kerr-AdS black hole - is stable itself (Holzegel-Smulevici 2013)

Nonlinear waves on bounded domains

- To gain insight into the dynamics of asymptotically AdS spacetimes it seems instructive to look at much simpler nonlinear wave equations on spatially bounded domains
- Example: $u_{tt} - \Delta u + u^3 = 0$ for $u(t,x)$ with $x \in M$ (compact manifold)
- Due to the lack of dispersion the long-time dynamics is much more complex and mathematically challenging than in the non-compact setting
- Is the ground state $u = 0$ stable (say in H_2 norm)?
- This is an open problem even for $u_{tt} - u_{xx} + u^3 = 0$ on S^1 **!**
- Key enemy: weak turbulence - transfer of energy to progressively smaller scales (gradual loss of smoothness as $t \to \infty$)
- Over the past few years the study of nonlinear wave equations on compact domains has become an active research direction in PDEs. The main goal is to understand out-of-equilibrium dynamics of small solutions.

General strategy for small initial data

- Let $e_k(x)$ and ω_k^2 be the eigenfunctions and eigenvalues of $-\Delta$ on M
- Decompose $u(t,x) = \varepsilon \sum_k a_k(t) e_k(x)$ and rewrite the equation on the Fourier side as an infinite dimensional dynamical system

$$\ddot{a}_n + \omega_n^2 a_n = \varepsilon^2 \sum c_{jkm}^n a_j a_k a_m, \qquad c_{jkm}^n = (e_j e_k e_m, e_n)$$

- The entire information about the dynamics is contained in the frequencies ω_n and the interaction coefficients c_{jkm}^n
- Are there non-trivial resonances?
  - If not: try to construct the solutions perturbatively (for example, using the method of normal forms). Main difficulty: small divisors.
  - If yes: drop all non-resonant terms and hope that the remaining resonant system is amenable to mathematical analysis
- Key object of interest: evolution of the energy spectrum $E_n(t) = \dot{a}_n^2 + \omega_n^2 a_n^2$. The transfer of energy to high fequencies can be measured by Sobolev norms $\|u(t)\|_s = \left(\sum \omega_n^{2s} a_n^2\right)^{1/2}$ with $s > 1$.

Example: cubic Klein-Gordon equation on S^1

- Plugging $u(t,x) = \varepsilon \sum_{n\in\mathbb{Z}} a_n(t) e^{inx}$ into $u_{tt} - u_{xx} + \mu^2 u + |u|^2 u = 0$ gives

$$\ddot{a}_n + \omega_n^2 a_n = -\varepsilon^2 \sum_{j-k+m=n} a_j \bar{a}_k a_m$$

- Interaction picture (variation of constants)

$$a_n = a_n^+(t) e^{i\omega_n t} + a_n^-(t) e^{-i\omega_n t}, \quad \dot{a}_n = i\omega_n \left(a_n^+(t) e^{i\omega_n t} - a_n^-(t) e^{-i\omega_n t} \right)$$

leads to the first order system ($\Omega = \pm\omega_j \pm \omega_k \pm \omega_m \mp \omega_n$)

$$\pm 2in\, \dot{a}_n^\pm = \varepsilon^2 \sum_{\substack{j-k+m=n \\ \text{permutations of } \pm}} a_j^\pm \bar{a}_k^\pm a_m^\pm e^{i\Omega t}$$

- Resonant terms correspond to $\Omega = 0$ and $j - k + m = n$. For nonzero mass μ there are no exact resonances ($\omega_n = \sqrt{n^2 + \mu^2}$). For $\mu = 0$, after dropping all non-resonant terms, one gets the resonant system

$$\pm 2in\, \dot{a}_n^\pm = \varepsilon^2 \sum_{j-k+m=n} a_j^\pm \bar{a}_k^\pm a_m^\pm + 2\varepsilon^2 \left(\sum_k |a_k^\mp|^2 \right) a_n^\pm$$

Numerical results

- We solve numerically $u_{tt} - u_{xx} + \mu^2 u + u^3 = 0$ on the interval $-1 \leq x \leq 1$ with periodic boundary conditions for different initial data ‣ Start movie
- For (small) initial data, after a very short time we observe the formation of a coherent structure with the exponentially decaying energy spectrum $E_k(t) \sim e^{-\alpha(t)k}$. The radius of analyticity $\alpha(t)$ quickly stabilizes at some (approximately) constant value (the Sobolev norms stop growing)
- Suprisingly, the dynamics for $\mu = 0$ and $\mu \neq 0$ are similar ‣ Start movie
- Analogous behaviour in higher dimensions ‣ Start movie
- It is conceivable that this coherent structure is a transient metastable state with an extremely long lifetime (as in the Fermi-Pasta-Ulam system)
- What is the mechanism of the saturation of the energy transfer?

Yang-Mills on the Einstein universe

- Manifold $M = \mathbb{R} \times S^3$ with the metric

$$g = -dt^2 + dx^2 + \sin^2 x \, (d\vartheta^2 + \sin^2\vartheta \, d\varphi^2)$$

- Equivariant (magnetic) ansatz for the SU(2) Yang-Mills connection

$$A = W(t,x)\tau_1 d\vartheta + (\cot\vartheta \, \tau_3 + W(t,x)\tau_2) \sin\vartheta \, d\varphi$$

- The YM equations $\nabla_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0$ reduce to

$$\boxed{W_{tt} = W_{xx} + \frac{W(1-W^2)}{\sin^2 x}}$$

- For smooth initial data the solutions remain smooth for all times (Choquet-Bruhat 1989, Chruściel-Shatah 1997)

- The conserved energy $E = \int_0^\pi \left(W_t^2 + W_x^2 + \frac{(1-W^2)^2}{2\sin^2 x} \right) dx$
- $W(t,0) = \pm 1$ and $W(t,\pi) = \pm 1 \Rightarrow$ two topological sectors $N = 0, 1$.
- In each sector there is a unique static solution:
  $W_0 = 1$ (vacuum) with $E = 0$ and $W_1 = \cos x$ (kink) with $E = 3\pi/4$.
- Linearized perturbations $u = W - W_N$ around the static solutions satisfy

  $$u_{tt} + Lu = 0\,, \qquad L = -\frac{d^2}{dx^2} + \frac{3W_N^2 - 1}{\sin^2 x}$$

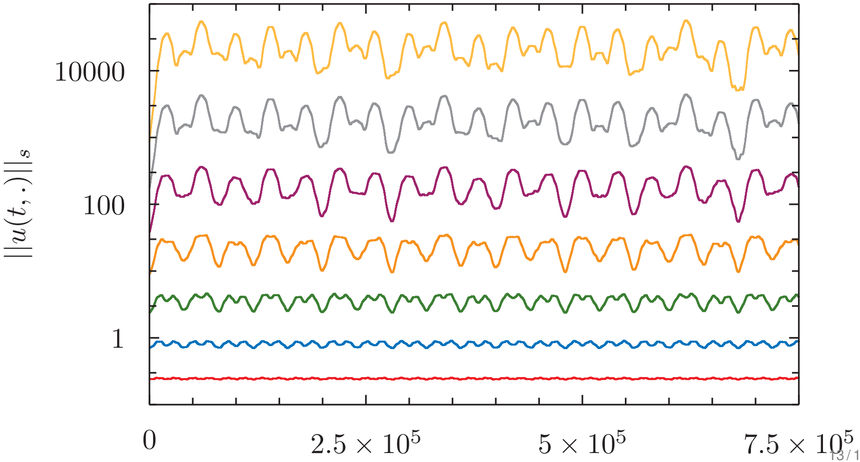
  The operator L is essentially self-adjoint on $L^2([0,\pi], dx)$.
- The eigenvalues and (orthonormal) eigenfunctions of L are $(k = 0, 1, ...)$

  $$\omega_k^2 = (2+k)^2 \quad \text{for} \quad N = 0 \quad \text{and} \quad \omega_k^2 = (2+k)^2 - 3 \quad \text{for} \quad N = 1$$

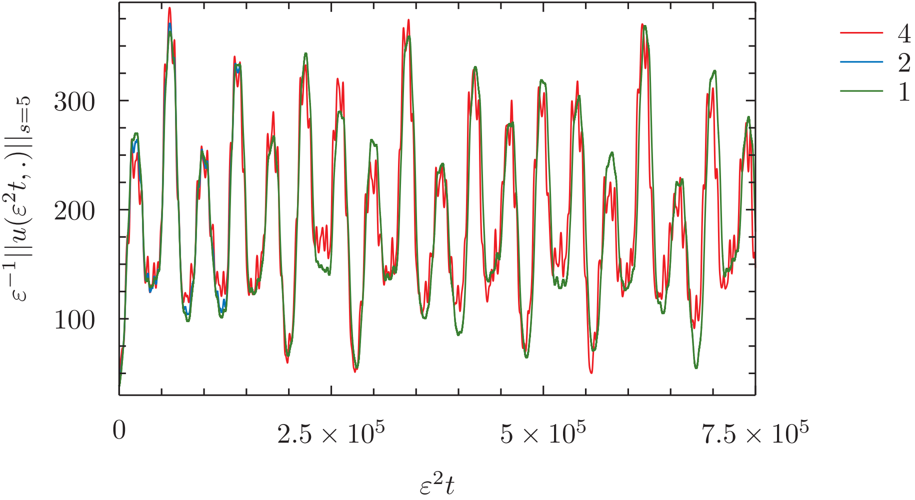
  $$e_0 = \sqrt{\tfrac{8}{3\pi}}\,\sin^2 x,\ e_1 = \sqrt{\tfrac{16}{\pi}}\,\sin^2 x \cos x,\ e_2 = \sqrt{\tfrac{32}{15\pi}}\,\sin^2 x\,(6\cos^2 x - 1), ...$$

Numerical results

- Transfer of energy to high frequencies is much more effective in the fully resonant case, yet (in both cases) the energy spectrum gets frozen after some time Start movie
- Sobolev norms ($s = 1, ..., 7$) for a gaussian perturbation of W_0



Evidence for (meta)stability of W_0



The scaling of $||u(t)||_5$ with respect to the amplitude ε of the gaussian.

Comment on AdS boundary conditions for Yang-Mills

- AdS is conformal to half of the Einstein universe ($0 \le x \le \frac{\pi}{2}$). Since the YM equations are conformally invariant in four dimensions, they are the same on AdS and the Einstein universe
- Restricting the solution of YM equations on the Einstein universe to the northern hemisphere one gets the solution of YM equations in AdS (with some complicated time-dependent boundary conditions)
- The AdS boundary $x = \frac{\pi}{2}$ is regular for the YM equations and consequently there is a huge freedom of prescribing the boundary conditions (cf. Friedrich 2014): not only Dirichlet, Neumann, and Robin, but also **energy non-conserving boundary conditions**.
- For example, one can impose the ”outgoing wave condition” $W_t + W_x = 0$ at $x = \frac{\pi}{2}$. Then $\frac{dE}{dt} = -W_t^2(t, \frac{\pi}{2})$, hence the energy leaks out from AdS.
- For the same reason **it is very easy to grow YM hair on AdS (and AdS black holes) in four dimensions**: almost any static solution that is good at the origin (or at the horizon) is good at $x = \frac{\pi}{2}$ as well. The question of linear stability of such solutions is inherently ambiguous.

Equivariant wave maps from $\mathbb{R} \times S^3$ into S^3

$$U_{tt} = U_{xx} + 2\cot x \, U_x - \frac{\ell(\ell+1)}{2} \frac{\sin(2U)}{\sin^2 x}$$

- Infinitely many static solutions (harmonic maps between 3-spheres)
- Blowup for large data is governed by self-similar wave maps from Minkowski space into S^3 (blowup does not see the curvature)
- Is there a threshold for blowup? One may speculate that the lack of dispersion combined with the supercritical scaling of energy can lead to blowup for arbitrarily small perturbations (as in the case of AdS)
- Numerical simulations for initial data $U(0,x) = \varepsilon f(x)$ indicate that there is a decreasing sequence of critical amplitudes ε_n for which the solution blows up at one of the poles (along the unstable self-similar solution), however this sequence accumulates strictly above zero.
- This toy model has one drawback: the linear spectrum is not resonant

Conclusions

- Dynamics of asymptotically AdS spacetimes is an interesting meeting point of basic problems in general relativity and PDE theory.
- Understanding of the out-of-equilibrium dynamics of small solutions is mathematically challenging even for the simplest nonlinear wave equations on compact manifolds, let alone Einstein's equations.
- The above simple models exhibit a qualitatively different behaviour than Einstein-AdS equations - in this sense they are not good toy models.
- Yet, such studies are instructive as they help us to understand how special Einstein's equations are (and they are interesting on their own).
- Keep searching for better toy models: supercritical semilinear wave equations with a fully resonant linear spectrum, quasilinear equations?